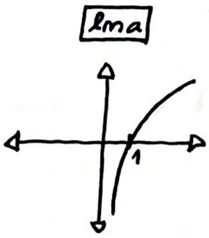
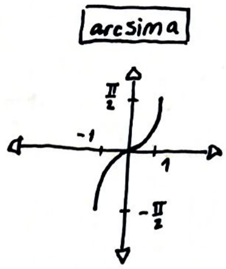
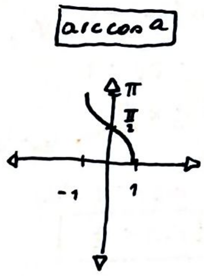
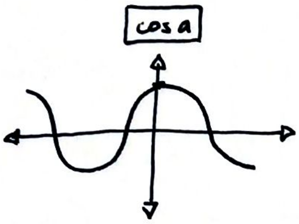
① DEFINICIJSKO OBMOČJE

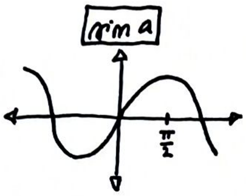
- ulomki : imenovalec $\neq 0$
- $\ln a \rightarrow a > 0$
- $\log a \rightarrow a > 0$
- $\sqrt{a} \rightarrow a \geq 0$

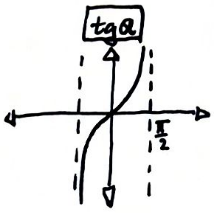


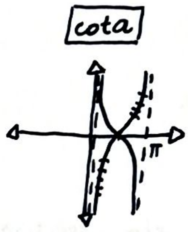




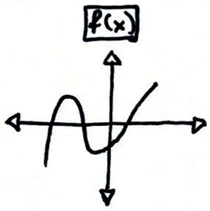


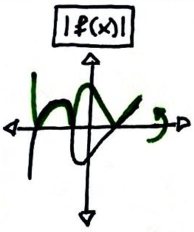


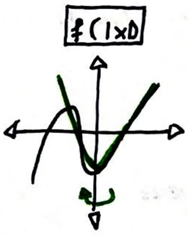




ABSOLUTNA VREDNOST







$D_f = D_g \cap D_h$

$D_h = I_1 \cup I_2$

1) $\ln(1 - |2 - |x-1||)$

① $1 - |2 - |x-1|| > 0$

$|2 - |x-1|| < 1$ absolutno na svojo stran, številke na svojo stran

intervali

② $x \geq 1$

$x - 1 \geq 0$ ⊕

$|x-1| = x-1$ ⊕

$x < 1$

$x - 1 < 0$ ⊖

$|x-1| = -(x-1)$ ⊖

③ $|2-(x-1)| < 1$

$|3-x| < 1$

$-1 < 3-x < 1 \; /-3$

$-4 < -x < -2 \; /\cdot(-1)$

$\boxed{2 < x < 4}$

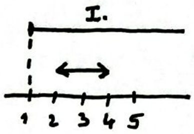
$|2+(x-1)| < 1$

$|x+1| < 1$

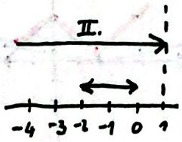
$-1 < x+1 < 1 \; /-1$

$\boxed{-2 < x < 0}$

④



$I_1 = (2,4)$



$I_2 = (-2,0)$

⑤ $D_f = I_1 \cup I_2 = (-2,0) \cup (2,4)$

2.) $\ln(|x^2 - 5x| - 6)$

① $|x^2 - 5x| - 6 > 0$

$|x^2 - 5x| > 6$

$|x(x-5)| > 6$

⊕ ⊖

poiskus z vstavljanjem številk

② $x(x-5) \geq 0$

$x_1 = 0$ $\quad x_1 < 0$

$x_2 = 5$ $\quad x_2 \geq 5$

$x(x-5) < 0$

$x_1 = 0$ $\quad x_1 > 0$

$x_2 = 5$ $\quad x_2 < 5$

③ $x^2 - 5x > 6$

$x^2 - 5x - 6 > 0$

$(x-6)(x+1) > 0$

$x_1 = 6$ $\quad x_1 > 6$

$x_2 = -1$ $\quad x_2 < -1$

$-x^2 + 5x > 6$

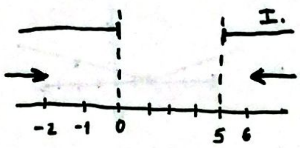
$-x^2 + 5x - 6 > 0$

$(x-3)(x-2) < 0$

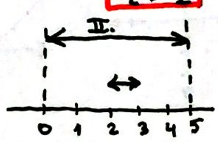
$x_1 = 3$ $\quad x_1 < 3$

$x_2 = 2$ $\quad x_2 > 2$

④



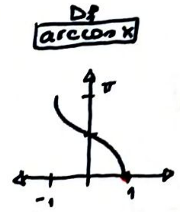
$I_1 = (-\infty,-2) \cup (6,\infty)$

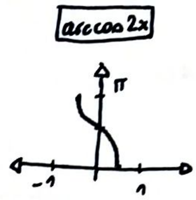


$I_2 = (2,3)$

$D_f = I_1 \cup I_2 = (-\infty,-2) \cup (2,3) \cup (6,\infty)$

3) $f(x) = \ln(\arccos(2x))$





$\arccos(2x) > 0$

za x

$-1 \leq Df < 1$

$-1 \leq x < 1$ ker je vrednost $\emptyset$

za 2x

$-1 < 2x < 1 \quad /:2$

$\boxed{-\frac{1}{2} \leq x < \frac{1}{2}}$

INJEKTIVNOST, SURJEKTIVNOST, BIJEKTIVNOST

INJEKTIVNOST

graf funkcije seka vzporednico x-osi 1 ali NOBENKRAT

SURJEKTIVNOST

graf funkcije seka vzporednico x-osi 1 ali VEČKRAT (vsaj 1)

BIJEKTIVNOST

graf funkcije seka vzporednico x-osi 1

samo če je funkcija bijektivna lahko delamo INVERZ

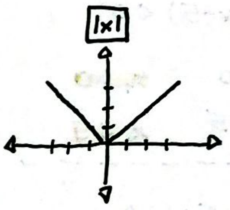
1) $f(x) \rightarrow y$
2) zamenjaš x in y
3) izpostaviš y
4) $y \rightarrow f^{-1}(x)$

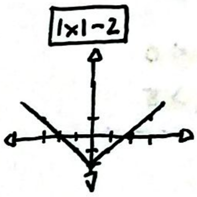
NEENAKOST

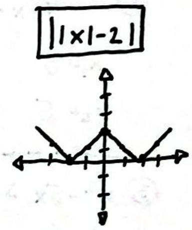
1.) $||x|-2| < \frac{1}{4}|x+2|$

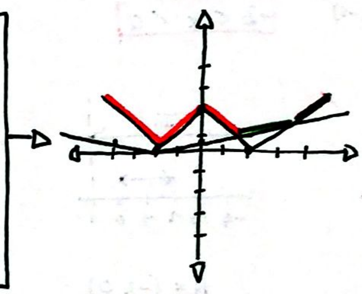
① GRAFIČNO

~~Vsak~~ $||x|-2|$ (od znotraj navzven)

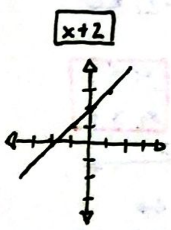


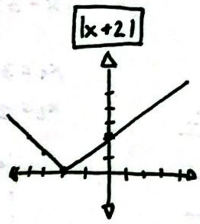


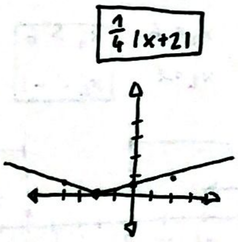




$\frac{1}{4}|x+2|$







② PRESEČIŠČE

③ interval

④ ŠTEVILA

OMEJENOST

- $\alpha \in \mathbb{R}$ im je NATANČNA zg. MEJA / SUPREMUM ($\alpha = \sup A$), če:
  - α je zg. meja množice A ($a \leq \alpha$, $\forall a \in A$)
  - α je najmanjša zg. meja
- $\alpha \in \mathbb{R}$ im je NATANČNA SPODNJA MEJA / INFIMUM ($\alpha = \inf A$), če:
  - α je sp. MEJA
  - α je največja sp. MEJA

$$A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$$

$\sup A = 1 = \max A$ (najmanjša zg. meja - leži v množici)

$\inf A = 0$ $\quad$ $\min A$ = ne obstaja (ker ne obstaja v množici $\mathbb{N}$)

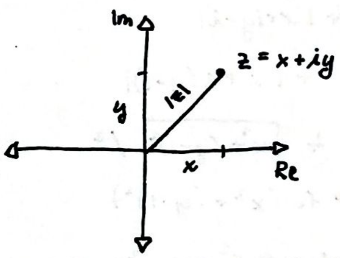
ne rabita ležati v isti množici

KOMPLEKSNA ŠTEVILA

- zapis: $z = x + iy$ $\quad$ ($x, y \in \mathbb{R}$)

  $x = \operatorname{Re} z$

  $y = \operatorname{Im} z$



PRAVILA ZA RAČUNANJE $\mathbb{C}$

- $\bar{z} = x - iy$ konjugiramo $\mathbb{C}$ št.
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{(\bar{z})} = z$
- $z \cdot \bar{z} = x^2 + y^2$
- $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}$ absolutna vrednost $\mathbb{C}$
- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $|\bar{z}| = z$
- $|z_1 + z_2| \leq |z_1| + |z_2|$

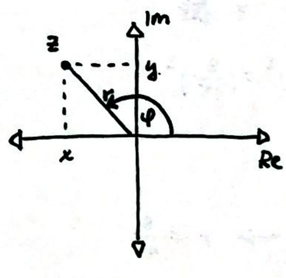
- $z^{-1} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2}$

* pri $\mathbb{C}$ št. vedno najprej razširi ulomek (da dobiš razliko kvadratov v imenovalcu)

POLARNI ZAPIS

- $z = x + iy = r(\cos\varphi + i \sin\varphi)$
- $z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2))$
- $z^n = r^n (\cos(n\varphi) + i \sin(n\varphi))$ $\quad n \in \mathbb{N}$

  DE MOIVREOVA FORMULA



* φ VEDNO V RAD $\quad$ * VEDNO NARIŠI SKICO! $\quad$ * namesto z piši $z = x + iy$

1) $2|z| = |z-3i|$

$z = x + iy$ $\quad$ $z = x+iy$

$|z| = \sqrt{x^2+y^2}$ $\quad$ $|(x+iy-3)|$

$2\sqrt{x^2+y^2} = \sqrt{x^2+(y-3)^2} \;/^2$

$4(x^2+y^2) = x^2+(y-3)^2$

$4x^2+4y^2 = x^2+y^2-6y+9$

$3x^2+3y^2+6y-9=0 \;/:3$

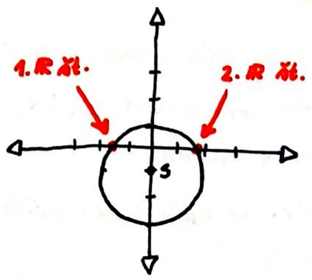
$x^2+y^2+2y-3=0$ DOPOLNJEVANJE DO POPOLNEGA KVADRATA

$x^2+(y+1)^2-1-3=0$

$x^2+(y+1)^2=4$

KROŽNICA: $x^2+y^2=r^2 \rightarrow r=2$

$S(0,-1)$



KOMPLEKSNO ŠT. JE ℝ TAKRAT, ko je $y=0$

$x^2+(y+1)^2=4$

$x^2+(0+1)^2=4$

$x^2+1=4$

$x^2=3$

$x=\pm\sqrt{3}$ $\rightarrow$ $x_1=\sqrt{3}$, $x_2=-\sqrt{3}$

RAZDALJA MED ℝ številoma $(0,\sqrt{3})$ in $(0,-\sqrt{3})$ je $2\sqrt{3}$

2) $\left|\frac{z+i}{z-i}\right| = 4 \;/\cdot|z-i|$

$|z+i| = 4|z-i|$ $\quad$ $z=x+iy$

$|x+iy+i| = 4|x+iy-i|$

$|x+i(y+1)| = 4|x+i(y-1)|$

$\sqrt{x^2+(y+1)^2} = 4\sqrt{x^2+(y-1)^2} \;/^2$

$x^2+(y+1)^2 = 16(x^2+(y-1)^2)$

$x^2+y^2+2y+1 = 16x^2+16y^2-32y+16$

$15x^2+15y^2-34y+15=0 \;/:15$

$x^2+y^2-\frac{34}{15}y+1=0$

$x^2+(y^2-\frac{34}{15}y)+1=0$

$x^2+(y-\frac{17}{15})^2-(\frac{17}{15})^2+1=0$

$x^2+(y-\frac{17}{15})^2-\frac{289}{225}+\frac{225}{225}=0$

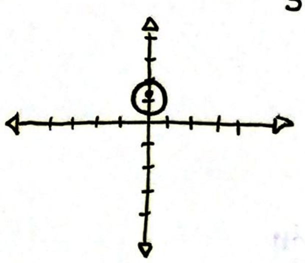
$x^2+(y-\frac{17}{15})^2=\frac{64}{225}$

$x^2+(y-\frac{17}{15})^2=(\frac{8}{15})^2$

KROŽNICA

$x^2+y^2=r^2 \rightarrow r=\frac{8}{15}$

$S(0,\frac{17}{15})$

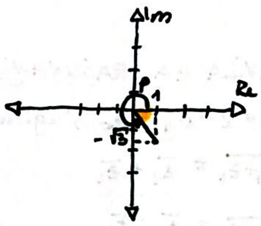


3) POLARNA OBLIKA

$z_1 = 1-\sqrt{3}i$

$z_1 = r(\cos\varphi + i\sin\varphi)$

$z_1 = 2\cdot(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3})$



$r^2=x^2+y^2$

$r^2=1+(-\sqrt{3})^2$

$r^2=1+3$

$r=\sqrt{4}=2$

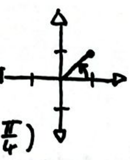
$\varphi = \cancel{360°} - \alpha$

$\alpha \Rightarrow tg\,\alpha = \frac{\sqrt{3}}{1} = \frac{\pi}{3} = \alpha$

$\varphi = \frac{5\pi}{3}$

$z_2 = 1+i$

$z_2 = r(\cos\varphi - i\sin\varphi)$



$z_2 = \sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})$

$$\frac{z_1^{12}}{z_2^{12}} = \frac{2^{12}\cdot(\cos(12\cdot\frac{5\pi}{3})+i\sin(12\cdot\frac{5\pi}{3}))}{\sqrt{2}^{12}(\cos(12\,\frac{\pi}{4})+i\sin(12\,\frac{\pi}{4}))}$$

$$= 2^{6}\,\frac{\cos(20\pi)+i\sin(20\pi)}{\cos(3\pi)+i\sin(3\pi)}$$

$$= 2^{6}\,\frac{1+i\cdot 0}{-1+i\cdot 0}$$

$$= 2^{6}\,\frac{1}{-1} = -2^{6}$$

4) $\sqrt{2}\,z^2 = (\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)^{21}+\frac{\sqrt{2}}{2}$

$(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i)^{21} = \cos 21\cdot\frac{7\pi}{4} - i\sin 21\cdot\frac{7\pi}{4} = \cos\frac{147\pi}{4} - i\sin\frac{147\pi}{4}$

5. VEKTORJI

= USMERJENA DALJICA S SMERJO, DOLŽINO IN USMERJENOSTJO

OPERACIJE

VSOTA VEKTORJEV

$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

$\vec{a} = (1, 2, 3)$

$\vec{b} = (-3, 7, 5)$

$\vec{a}+\vec{b} = (1,2,3) + (-3,7,5)$

$= (1+(-3),\ 2+7,\ 3+5) = (-2, 9, 8)$ KOMPONENTE

MNOŽENJE VEKTORJA S SKALARJEM

$\vec{a} = (1,2,3)$

$\alpha \cdot \vec{a} = \alpha(1,2,3) = (\alpha, 2\alpha, 3\alpha)$

$3 \cdot \vec{a} = 3(1,2,3) = (3, 6, 9)$

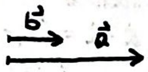
dolžina: $|\lambda| \cdot \|\vec{a}\|$

$\lambda > 0 \Rightarrow$ ohrani smer

$\lambda < 0 \Rightarrow$ spremeni smer

LINEARNA KOMBINACIJA VEKTORJEV

- $\vec{b} = \alpha\vec{a} \Leftarrow \vec{b} \parallel \vec{a}$ KOLINEARNA



  $\Rightarrow \alpha\vec{a} + \beta\vec{b} = \vec{0}$ $(\alpha, \beta \neq 0)$

- $\vec{c} = \alpha\vec{a} + \beta\vec{b} \Rightarrow$ KOMPLANARNA (= z dvema baznima vektorjema lahko izrazimo 3 vektor)

  $\Rightarrow \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0}$ $(\alpha, \beta, \gamma \neq 0) \rightarrow$ vektorji $\vec{a}, \vec{b}, \vec{c}$ so LINEARNO NEODVISNI

  $\Rightarrow$ so BAZE (nardi preizkus za vse komponente!)

- 3 linearno neodvisni vektorji tvorijo bazo prostora $\{\vec{a}, \vec{b}, \vec{c}\}$

  $\rightarrow$ vsak vektor lahko zapišemo z baznimi vektorji: $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$

  ↳ α, β, γ ... koordinate vektorja $\vec{d}$

  $\vec{a}, \vec{b}, \vec{c}$ ... glede na bazo $\vec{a}, \vec{b}, \vec{c}$

$\vec{a} = 3\vec{i} + 5\vec{j} - 7\vec{k} \rightarrow \vec{a} = (3, 5, -7)$

$\vec{b} = 2\vec{i} - 3\vec{j} + 2\vec{k} \rightarrow \vec{b} = (2, -3, 2)$

$\vec{a}+\vec{b} = \vec{i}\,(3+2) + \vec{j}\,(5-3) + \vec{k}\,(-7+2) = 5\vec{i} + 2\vec{j} - 5\vec{k} \rightarrow \vec{a}+\vec{b} = (5, 2, -5)$

BAZA → ORTOGONALNA (pravokotna $\vec{a}\perp\vec{b}, \vec{b}\perp\vec{c}, \vec{a}\perp\vec{c}$)

BAZA → NORMIRANA (enake dolžine $\|\vec{a}\| = \|\vec{b}\| = \|\vec{c}\|$)

} ORTONORMIRANA BAZA (ON)

npr. standardna baza prostora $\mathbb{R}^3$

$\vec{i} = (1, 0, 0)$

$\vec{j} = (0, 1, 0)$

$\vec{k} = (0, 0, 1)$

- KRAJEVNI VEKTORJI (od izhodišča do točke)

MNOŽENJE KOMPONENT

$\vec{a} = (1,2,3)$

$\vec{b} = (-3,7,5)$

$\vec{a} \cdot \vec{b} = (1,2,3) \cdot (-3,7,5)$

$= 1\cdot(-3) + 2\cdot 7 + 3 \cdot 5$

$= -3 + 14 + 15 = 26$ SKALAR

dolžina $\vec{b}$

$(3\vec{a}+\vec{b})(\vec{a}+\vec{b}) = 3\vec{a}\vec{a} + 3\vec{a}\vec{b} + \vec{b}\vec{a} + \vec{b}\vec{b} = 3\|\vec{a}\|^2 + \underbrace{4\vec{a}\vec{b}}_{} + \|\vec{b}\|^2$

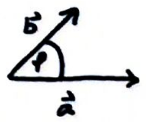
dolžina $\vec{a}$

skalarni produkt: $4 \cdot \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\varphi$

SKALARNI PRODUKT

$$\boxed{\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\varphi}$$

Skalarni produkt 2 vektorjev je ŠTEVILO



$\vec{a} = (1,2,3)$

$\vec{b} = (2,2,2)$

$\varphi = 22{,}2°$

$\vec{a}\cdot\vec{b} = (1,2,3)\cdot(2,2,2)$
$= 1\cdot2 + 2\cdot2 + 3\cdot2$
$= 2+4+6$
$= 12$

DOLŽINA VEKTORJA

$$\boxed{\|\vec{a}\| = \sqrt{a_1^2+a_2^2+a_3^2}}$$

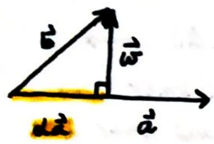
$\|\vec{a}\| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$\|\vec{b}\| = \sqrt{2^2+2^2+2^2} = \sqrt{4+4+4} = \sqrt{12}$

$\|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\varphi = \sqrt{14}\cdot\sqrt{12}\cdot\cos 22{,}2° = 12$

$\vec{a}\cdot\vec{b} = 0 \Rightarrow \varphi = 90°$

PRAVOKOTNA PROJEKCIJA



$\vec{a} = (6,-3,2)$

$\vec{b} = (2,-2,2)$

$$\boxed{\vec{b} = \alpha\vec{a} + \vec{w}}$$

$\vec{a}\cdot\vec{b} = \vec{a}\cdot(\alpha\vec{a}+\vec{w}) = \alpha\vec{a}\cdot\vec{a} + \vec{w}\cdot\vec{a}$; $\vec{w}\cdot\vec{a}$: 0 ker $\vec{w}\perp\vec{a}$

$\Rightarrow \alpha = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \dfrac{14}{49} = \dfrac{2}{7}$

$\vec{a}\cdot\vec{b} = (6,-3,2)\cdot(2,-2,2)$
$= 12+6+4 = 14$

$\vec{a}\cdot\vec{a} = (6,-3,2)\cdot(6,-3,2)$
$= (36+9+4) = 49$

DOLŽINA PRAVOKOTNE P.

$$\boxed{\|\alpha\vec{a}\| = \frac{|\vec{a}\cdot\vec{b}|}{\|\vec{a}\|}}$$

$$\boxed{\alpha = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}}$$

$\vec{b} = \frac{2}{7}\vec{a} + \vec{w}$

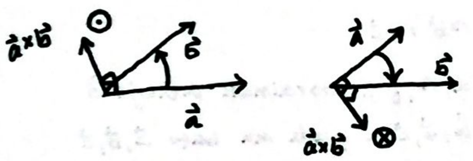
$\Rightarrow \vec{w} = \vec{b} - \frac{2}{7}\vec{a} = (2,-2,-2) - \left(\frac{12}{7}, -\frac{6}{7}, \frac{4}{7}\right) \Rightarrow \vec{w} = \frac{2}{7}(1,-4,-9)$

VEKTORSKI PRODUKT

$$\boxed{\|\vec{a}\times\vec{x}\| = \|\vec{a}\|\|\vec{x}\|\sin\varphi}$$ geometrijsko ploščina PARALELOGRAMA

$\left.\begin{array}{l}\vec{a}\times\vec{b}\perp\vec{a}\\ \vec{a}\times\vec{b}\perp\vec{b}\end{array}\right\}$ urejena baza

Vektorski produkt 2 vektorjev je VEKTOR



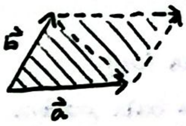
$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ ZAPOREDJE!

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -2 & 1 & 2 \end{vmatrix} =$

$= \vec{i}\,(2\cdot2-1\cdot3) - \vec{j}\,(1\cdot2-3\cdot(-2)) + \vec{k}\,(1\cdot1-2\cdot(-2))$

$= (1,-8,5)$

$\vec{a}\times\vec{b} = 0 \Rightarrow \varphi = 0°$ $(\vec{a}\parallel\vec{b})$



$\|\vec{a}\times\vec{b}\| = pl_{\square}$

$pl_{\square} = \|(1,-8,5)\| = \sqrt{1^2+(-8)^2+5^2} = \sqrt{1+64+25} = \sqrt{90} = 3\sqrt{10}$

$\vec{b}\times(\vec{a}+\vec{b}) = \vec{b}\times\vec{a} + \cancel{\vec{b}\times\vec{b}}$ 0

$\vec{a} = (1,2,3)$

$\vec{b} = (-2,1,2)$

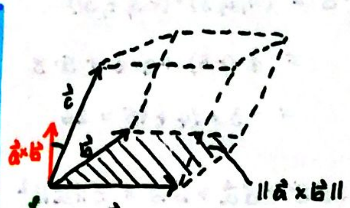
$$\vec{a}\times\vec{b} = \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{array} = \vec{i}\,(a_2b_3-b_2a_3) - \vec{j}\,(a_1b_3-b_1a_3) + \vec{k}\,(a_1b_2-a_2b_1)$$

MEŠANI PRODUKT

mešani produkt 3 vektorjev je SKALAR

$$[\vec{a},\vec{b},\vec{c}] = \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{array}$$

$$= a_1(b_2c_3-c_2b_3) - a_2(b_1c_3-b_3c_1) + a_3(b_1c_2-b_2c_1)$$



$V_{\square} = |[\vec{a},\vec{b},\vec{c}]|$

PARALEPIPED

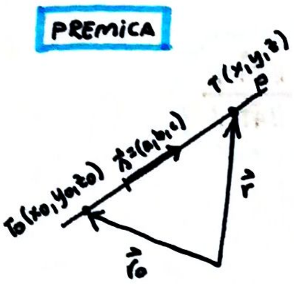
6 ANALITIČNA GEOMETRIJA z VEKTORJI v $\mathbb{R}^3$

TOČKA

$\vec{r}_T = \vec{OT} = \vec{i}x + \vec{j}y + \vec{k}z \rightarrow T(x,y,z)$

podana je s **krajevnim vektorjem**

PREMICA



① VEKTORSKA ENAČBA PREMICE

$\boxed{\vec{r} = \vec{r_0} + \lambda \vec{s}}$ $\quad \lambda \in \mathbb{R}$

$(x,y,z) = (x_0,y_0,z_0) + (\lambda a, \lambda b, \lambda c)$

$(x,y,z) = (x_0 + \lambda a,\ y_0 + \lambda b,\ z_0 + \lambda c)$

premica je podana s **točko** $T_0 \in p$ in **smernim vektorjem** $\vec{s} \neq 0$ (ki mu je vzporedna)

ALI

z **2 TOČKAMA**

$\vec{s}_p = \vec{s}_q$

$\Rightarrow p \parallel q$

② PARAMETRIČNA ENAČBA P.

$$\left.\begin{aligned} x &= x_0 + \lambda a \\ y &= y_0 + \lambda b \\ z &= z_0 + \lambda c \end{aligned}\right\} \lambda \in \mathbb{R} \longrightarrow$$

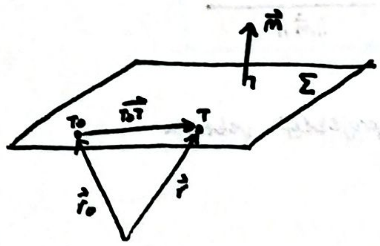
③ KANONIČNA ENAČBA P.

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$

PRESEČIŠČE 2 PREMIC

iz p izraziš λ

RAVNINA



ravnina je določena z:

① **1 točko** $T_0(x_0,y_0,z_0)$ in **normalnim vektorjem** $\vec{n} = (a,b,c) \neq 0$ ki je na Σ pravokoten

$\vec{T_0T} \perp \vec{n} \Rightarrow \vec{T_0T} \cdot \vec{n} = 0$

$\boxed{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}$ VEKTORSKA ENAČBA R.

② $(x-x_0,\ y-y_0,\ z-z_0)(a,b,c) = 0$

$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}$

$= d \rightarrow d$ je neka številka, če je 0 gre ravnina skozi izhodišče

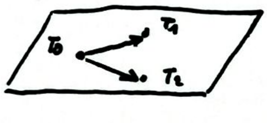
$\boxed{ax + by + cz = d}$ SPLOŠNA ENAČBA R.

③ namesto $\vec{n}$, lahko tudi **2 LINEARNO NEODVISNA VEKTORJA** ki ležita v ravnini

$\boxed{\vec{n} = \vec{n_1} \times \vec{n_2}}$

$\boxed{\vec{r} = \vec{r_0} + \vec{T_0T} = \vec{r_0} + \lambda \vec{s_1} + \mu \vec{s_2}}$ $\quad \lambda, \mu \in \mathbb{R}$

④ **3 NEKOLINEARNE TOČKE**

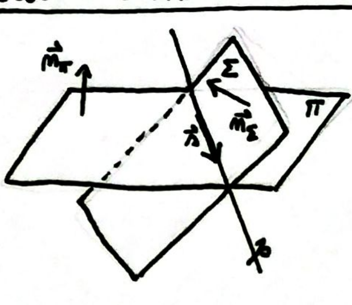


$\boxed{\vec{n} = \vec{T_0T_1} \times \vec{T_0T_2}}$

$\vec{n}_\Sigma \parallel \vec{n}_\Pi$

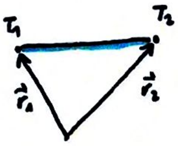
$\Rightarrow \Sigma \parallel \Pi \rightarrow$ v klasični enačbi se spremeni samo d

PRESEČIŠČE 2 RAVNIN $\rightarrow$ PREMICA



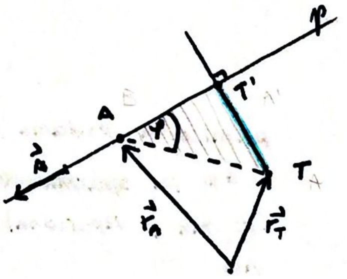
$$\left.\begin{aligned} \vec{s} &\perp \vec{n}_\Sigma \\ \vec{s} &\perp \vec{n}_\Pi \end{aligned}\right\} \vec{s} \Rightarrow \vec{s} \parallel \vec{n}_\Pi \times \vec{n}_\Sigma$$

RAZDALJA MED TOČKAMA



$$\boxed{d(T_1,T_2) = \|\overrightarrow{T_1T_2}\|}$$

RAZDALJA MED TOČKO IN PREMICO

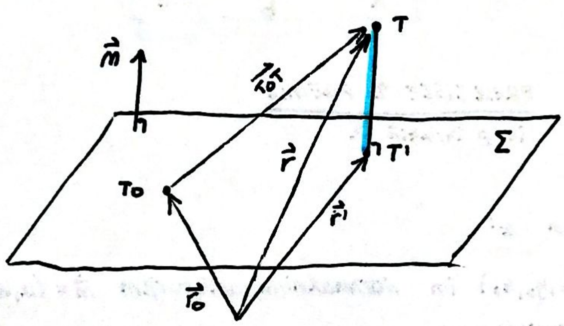


$$d(T,p) = \|\overrightarrow{TT'}\|$$

$$\boxed{d(T,p) = \frac{\|\overrightarrow{AT} \times \vec{s}\|}{\|\vec{s}\|}}$$

$$\sin\varphi = \frac{\|\overrightarrow{TT'}\|}{\|\overrightarrow{AT}\|}$$

RAZDALJA MED TOČKO IN RAVNINO



$$d(T,T') = \|\overrightarrow{TT'}\|$$

$$\vec{r}' = \vec{r} + \lambda \cdot \vec{n}$$

$$\boxed{d(T,T') = d(T,\Sigma) = \frac{|(\vec{r}_T - \vec{r}_0)\cdot\vec{n}|}{\|\vec{n}\|}}$$

dolžina $\overrightarrow{TT'}$ je enaka pravokotni projekciji vektorja $\overrightarrow{T_0T}$ na $\vec{n}$

7) MATRIKE

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \leftarrow \text{1. VRSTICA}$$

m ŠTEVILO VRSTIC

1. STOLPEC

n ŠTEVILO STOLPCOV

ENAKI MATRIKI

matriki $A=[a_{ij}]_{m\times n}$ in $B=[b_{ij}]_{p\times r}$ sta enaki, če sta:

1) ENAKIH DIMENZIJ (enako št. stolpcev in vrstic): $m=p$ in $n=r$
2) VSI ISTOLEŽEČI ELEMENTI MORAJO BITI ENAKI: $a_{ij}=b_{ij}$, $\forall i$ in j

POSEBNE MATRIKE

1) **NIČELNA MATRIKA**

$a_{ij}=0$

$$0=\underbrace{\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{n}\Big\}m \qquad m,n\in\mathbb{R}^{m\times n}$$

$0_{m\times n}$

2) **ZGORNJE TRIKOTNA**

$a_{ij}=0$ za $i>j$

$$A=\begin{bmatrix} a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&0&a_{33}\end{bmatrix} = \quad = \quad 0$$

3) **SPODNJE TRIKOTNA**

$a_{ij}=0$ za $i<j$

$$A=\begin{bmatrix} a_{11}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{bmatrix}$$

4) **KVADRATNA MATRIKA**

$A\in\mathbb{R}^{n\times n}$

5) **ENOTSKA M. (IDENTITETA)**

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

6) **DIAGONALNA**

$$D=\begin{bmatrix} d_{11}&&&0\\&d_{22}&&\\&&d_{33}&\\0&&&d_{44}\end{bmatrix}$$

7) **SIMETRIČNA**

$A\in\mathbb{R}^{n\times n}$

$a_{ji}=a_{ij}$, $\forall ij$

$$A=\begin{bmatrix}1&2&3\\2&5&6\\3&6&5\end{bmatrix}$$ SE PRESLIKAJO

OPERACIJE

1) **VSOTA**

A in $B\in\mathbb{R}^{m\times n}$ $\rightarrow$ ENAKIH DIMENZIJ!

SEŠTEVAMO PO KOMPONENTAH $\quad A+B=[a_{ij}+b_{ij}]_{m\times n}$

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \quad B=\begin{bmatrix}-1&0&3\\5&9&2\\7&-2&-5\end{bmatrix} \quad A+B=\begin{bmatrix}1+(-1)&2+0&3+3\\4+5&5+9&6+2\\7+7&8+(-2)&9+(-5)\end{bmatrix}=\begin{bmatrix}0&2&6\\9&14&8\\14&6&4\end{bmatrix}$$

2) **PRODUKT MATRIKE S SKALARJEM**

$\lambda\in\mathbb{R}$, $A\in\mathbb{R}^{m\times n}$

$\lambda\cdot A=[\lambda a_{ij}]_{m\times n}$

$$A=\begin{bmatrix}2&4\\3&7\end{bmatrix} \quad 3A=\begin{bmatrix}2\cdot3&4\cdot3\\3\cdot3&7\cdot3\end{bmatrix}=\begin{bmatrix}6&12\\9&21\end{bmatrix}$$

$\lambda=3$

VSAKO ŠTEVILKO POMNOŽIŠ S λ

LASTNOSTI VSOTA

1) $A+B=B+A$
2) $(A+B)+C = A+(B+C)$
3) $A+0=A$
4) $A+(-A)=0$

$-A=(-1)A$ nasprotna matrika

LASTNOSTI MNOŽENJE S SKALARJEM

1) $(\alpha+\beta)A = \alpha A + \beta A$
2) $\alpha(A+B) = \alpha A + \beta A$
3) $\alpha(\beta \cdot A) = (\alpha\beta)A$
4) $1\cdot A = A$

* OPOMBA

matrike $A \in \mathbb{R}^{m\times m}$ so vektorji (stolpci)

$A = \left.\begin{bmatrix} \vdots & \vdots \end{bmatrix}\right\}m$

$\vec{a} = (1,2,3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3. PRODUKT MATRIK

$A \in \mathbb{R}^{m\times n}$ in $B \in \mathbb{R}^{n\times p}$

$$A\cdot B = \left[\sum_{k=1}^{n} a_{ik}\cdot b_{kj}\right]_{m\times p}$$

$$A\cdot B = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot0+3\cdot1 & 2\cdot1+3\cdot0 & 2\cdot2+3\cdot1 \\ 1\cdot0+1\cdot1 & 1\cdot1+0\cdot1 & 1\cdot2+1\cdot1 \\ 2\cdot0+1\cdot1 & 2\cdot1+1\cdot0 & 2\cdot2+1\cdot1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix}$$

če z matriko množimo vektor $\rightarrow$ dobimo vektor

LASTNOSTI PRODUKTA MATRIK

naj bodo dimenzije matrik take, da produkt obstaja

1) $(A\cdot B)\cdot C = A\cdot(B\cdot C)$
2) $A\cdot I = A$, $I\cdot A = A$ množenje z identiteto (ne spremeni matrike)
3) $0\cdot A = 0$ (0 = številka ali matrika), $A\cdot 0 = 0$ (0 = obvezno matrika)
4) $(\alpha A)B = A\cdot(\alpha B) = \alpha(AB)$, $\alpha \in \mathbb{R}$
5) $(A+B)\cdot C = AC+BC$, $A(B+C) = AB+AC$
6) $BA \neq AB$

* OPOMBA

$A \in \mathbb{R}^{1\times 1}$

$A = [a_{11}]$

$[3] = 3$

matrika dimenzij 1×1 je realno število

4. TRANSPONIRANJE

$A \in \mathbb{R}^{m\times n}$ $\quad A^T \in \mathbb{R}^{n\times m}$

$(A^T)_{ij} = a_{ji}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

LASTNOSTI TRANSPONIRANJE

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(\alpha A)^T = \alpha A^T$
4) $(A\cdot B)^T = B^T A^T$

* OPOMBA

$A^k = \underbrace{A\cdot A\cdot \ldots \cdot A}_{k}$

$A^2 = A\cdot A$

$A^3 = A\cdot A\cdot A = A^2\cdot A$

SISTEM LINEARNIH ENAČB

ŠT. REŠITEV LINEARNIH ENAČB

$\emptyset \qquad 1 \qquad \infty$

(1.)
$$\begin{aligned} x+2y&=-4\\ 2x-y&=7\\ 7x+4y&=2\\ 5x+3y&=1 \end{aligned}$$

$$A=\begin{bmatrix}1&2\\2&-1\\7&4\\5&3\end{bmatrix} \qquad b=\begin{bmatrix}-4\\7\\2\\1\end{bmatrix} \qquad x=\begin{bmatrix}x\\y\end{bmatrix}$$

$A\cdot x=b \rightarrow$ želimo dobiti: $[A|b] \sim \left[\begin{array}{c|c} \diagdown & *\\ 0 & *\\ & * \end{array}\right]$

$$[A|b]=\left[\begin{array}{cc|c}1&2&-4\\2&-1&7\\7&4&2\\5&3&1\end{array}\right] \begin{matrix}/\cdot(-2)\ (-7)\ (-5)\end{matrix} \sim \left[\begin{array}{cc|c}1&2&-4\\0&-5&15\\0&-10&30\\0&-7&21\end{array}\right]\begin{matrix}\\ /:5\\ /:10\\ /:7\end{matrix} \sim \left[\begin{array}{cc|c}1&2&-4\\0&-1&3\\0&-1&3\\0&-1&3\end{array}\right] \sim \left[\begin{array}{cc|c}1&2&-4\\0&-1&3\\0&-1&3\\0&0&0\end{array}\right] \sim$$

3)

$$\sim \left[\begin{array}{cc|c}1&2&-4\\0&-1&3\\0&0&0\\0&0&0\end{array}\right] \begin{matrix}\rightarrow x+2y=-4\\ \rightarrow -y=3 \rightarrow \boxed{y=-3}\end{matrix}$$

$x+2(-3)=-4$

$\boxed{x=2}$

$$R=x=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\-3\end{bmatrix}$$

ENA REŠITEV

Sistem je rešljiv

→ če bi bila kakšna vrstica

npr. $[0\ \ 0\,|\,2]$ ne bi bilo rešitve saj $0x+0y\neq 2$ $[\ .\ *|0]$

NI REŠITVE

(2.) HOMOGEN SISTEM
$$\begin{aligned} x+2y+z+w&=0\\ 3x+6y-2z+w&=0\\ x+2y-4z-w&=0\\ 2x+4y-3z&=0 \end{aligned}$$

RAZŠIRJENA MATRIKA

2)

$$[A|b]=\left[\begin{array}{cccc|c}1&2&1&1&0\\3&6&-2&1&0\\1&2&-4&-1&0\\2&4&-3&0&0\end{array}\right] (-3)\ (-1)\ (-2) \sim \left[\begin{array}{cccc|c}1&2&1&1&0\\0&0&-5&-2&0\\0&0&-5&-2&0\\0&0&-5&-2&0\end{array}\right] (-1)\ (-1) \sim$$

$$\sim \left[\begin{array}{cccc|c}1&2&1&1&0\\0&0&-5&-2&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right] \begin{matrix}\rightarrow x+2y+z+w=0 \rightarrow x+2y+\beta-\frac{5}{2}\beta=0\\ \rightarrow -5z-2w=0\end{matrix}$$

$w=-\frac{5}{2}z$ $\qquad \boxed{z=\beta}$

$\boxed{w=-\frac{5}{2}\beta}$

$x=-2y+\frac{3}{2}\beta$ $\qquad \boxed{y=\alpha}$

$\boxed{x=-2\alpha+\frac{3}{2}\beta}$

$$x=\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-2\alpha+\frac{3}{2}\beta\\ \alpha\\ \beta\\ -\frac{5}{2}\beta\end{bmatrix}=\begin{bmatrix}-2\alpha\\ \alpha\\0\\0\end{bmatrix}+\begin{bmatrix}\frac{3}{2}\beta\\0\\ \beta\\ -\frac{5}{2}\beta\end{bmatrix}=\alpha\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+\beta\begin{bmatrix}\frac{3}{2}\\0\\1\\-\frac{5}{2}\end{bmatrix},\ \alpha \text{ in } \beta\in\mathbb{R}$$

VEČ REŠITEV

∞ mnogo rešitev

GAUSSOVA METODA

2 sistema linearnih enačb sta EKVIVALENTNA, če imata isto množico rešitev. Ideja Gaussove metode, je da se dani sistem $Ax=b$ prevede na ekvivalenten sistem $\tilde{A}x=\tilde{B}$, ki je v bolj enostavni (zg. trikotniški) obliki.

Elementarne operacije na vrsticah so:

1) ZAMENJAVA 2 VRSTIC
2) MNOŽENJE VRSTICE S SKALARJEM ($\neq 0$)
3) VRSTICI PRIŠTEJEMO DRUGO VRSTICO

Pri tem se množica rešitev ne spremeni. Operacije izvajamo na razširjeni matriki $[A|b]$

INVERZNA MATRIKA IN MATRIČNE ENAČBE

KVADRATNA MATRIKA A je OBRNLJIVA, če obstaja MATRIKA, ki jo označimo z A^{-1} (inverzna matrika ali obratna matrika matriki A)

$$\boxed{AA^{-1} = I = A^{-1}A}$$

Če inverzna matrika obstaja jo poiščemo z GAUSSOVIM POSTOPKOM

$$\boxed{[A|I] \sim [I|A^{-1}]}$$

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$$

$$[A|I] = \underbrace{\left[\begin{matrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{matrix}\right.}_{\text{matrika } A}\underbrace{\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]}_{\text{identiteta}} (-2)(-4) \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right] 1) \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{array}\right] (1) \sim$$

- matrika A

sedaj moramo na L strani dobiti identiteto, na D pa bo inverz

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & 6 & 1 & 1 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] (-2) \sim$$

$$\sim \underbrace{\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right.}_{\text{identiteta}}\underbrace{\left|\begin{matrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{matrix}\right]}_{\text{inverz } A^{-1}} = [I|A^{-1}] \qquad A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

če na levi strani dobiš vrstico $[0 \;\; 0 \;\; 0 \mid 8 \;\; 2 \;\; 3]$ inverz NE OBSTAJA!

MATRIČNE ENAČBE

najdi x

① $A^{-1} \backslash \; Ax = b$ (pazi! mi into kot $Ax = b \,/^{-1}$)

$\underbrace{A^{-1}A}_{I} x = A^{-1}b$

$x = A^{-1}b$

② $A^{-1} \backslash \; Ay + B = 0$

$\underbrace{A^{-1}A}_{I} y + A^{-1}B = 0$

$y + A^{-1}B = 0$

$y = -A^{-1}B$

③ $AX - 2X = A + I$

$(A - 2I)X = A + I$

pazi da dodaš še I

pazi da je X na levi

$(A-2I)^{-1} \backslash \; (A-2I)X = A + I$

$\underbrace{(A-2I)^{-1}(A-2I)}_{I} X = (A-2I)^{-1}(A+I)$

$X = (A-2I)^{-1}(A+I)$

④ $B - CB = 0$

$B(I - C) = 0$

⑤ $X = B^{-1}(A-I)^{-1}$

$X = ((A-I)B)^{-1}$

DETERMINANTE

= preslikava, ki KVADRATNI MATRIKI priredi $\mathbb{R}$ število. Definiramo jo rekurzivno

$\rightarrow m = 1$ $\quad A = [a_{11}]$ $\quad \det A = a_{11}$

$\rightarrow m \geq 2$ $\quad A = [a_{ij}]_{m \times m}$

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \qquad \det A = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{vmatrix} \overset{\text{RAZVOJ PO 3. STOLPCU}}{=} 0 \cdot \begin{vmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix} - 4 \cdot \begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 6 & 0 & 0 \end{vmatrix} +$$

$$+ 0 \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 6 & 0 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} \overset{\text{RAZVOJ PO 1. VRSTICI}}{=} -4 \cdot \left(0 \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 6 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & 2 \\ 6 & 0 \end{vmatrix} \right) = -4 \cdot 3 \cdot (0 \cdot 0 - 2 \cdot 6) = 144$$

<u>LASTNOSTI DETERMINANT</u> (vse kar velja za vrstice, velja tudi za stolpce)

1) če je A zgornje / spodnje trikotna $\rightarrow \det A =$ produkt diagonalnih elementov
2) če matriko B dobimo tako, da v matriki A zamenjamo 2 vrstici $\rightarrow \det B = -\det A$
   $\rightarrow$ če sta v matriki A 2 vrstici enaki je $\det A = 0$
3) če B dobimo tako, da v matriki A neko vrstico pomnožimo z nekim številom $\lambda \neq 0 \rightarrow$
   $\rightarrow \det B = \lambda \det A$
4) če B dobimo tako, da neki matriki A prištejemo linearno kombinacijo drugih vrstic matrike A $\rightarrow \det B = \det A$
   $\rightarrow$ če so vrstice linearno odvisne je $\det A = 0$
5) $\det A^T = \det A$
6) $\det(A \cdot B) = \det A \cdot \det B$, $\quad A, B \in \mathbb{R}^{m \times m}$
7) matrika A je obrnljiva, če $\det A \neq 0$
8) $\det(A + B) \neq \det A + \det B$
9) za $m \times m$ matriko A ~~velja~~ in $\alpha \in \mathbb{R}$ velja: $\det(\alpha A) = \alpha^m \det A$

DETERMINANTO LAHKO RAČUNAMO Z RAZVOJEM PO KATERIKOLI VRSTICI / KATEREMKOLI STOLPCU

LASTNE VREDNOSTI IN LASTNI VEKTORJI MATRIKE

Število λ je lastna vrednost matrike $A \in \mathbb{R}^{m \times m}$ če obstaja neničelni vektor $x \in \mathbb{R}^m$, da velja $Ax = \lambda x$. Vsak ta vektor x je lastni vektor, ki pripada lastni vrednosti λ

1) $\alpha \in \mathbb{R}, \alpha \neq 0$

   $AX = \lambda X \;/ \cdot \alpha$

   $\alpha A X = \alpha \lambda X$

   $A(\alpha X) = \lambda (\alpha X) \rightarrow \alpha X$ je tudi lastni vektor za λ = LASTNI VEKTOR NI ENOLIČNO DOLOČEN

2) lastne vrednosti matrike so lahko tudi KOMPLEKSNA ŠTEVILA

3) $AX = \lambda X \qquad x \neq 0$

   $AX - \lambda X = 0$

   $\underbrace{(A - \lambda I)}_{B} x = 0$

   ⊗ $BX = 0 \rightarrow$ sistem linearnih enačb (homogen) tak sistem ima vedno rešitev $x = 0$ ⊗

   ⊗ recimo, da je B obrnljiva $\rightarrow$ obstaja B^{-1}

   $B^{-1} / BX = 0$

   $\underbrace{B^{-1} B}_{I} x = 0$

   $I \quad x = 0$ ⊗

   ✓ $\rightarrow$ matrika $B = A - \lambda I$ ne sme biti obrnljiva

   KARAKTERISTIČNI POLINOM $\boxed{\det(A - \lambda I) = 0}$ ✓

   lastne vrednosti matrike A so NIČLE KARAKTERISTIČNEGA POLINOMA $p(\lambda)$. lastnih vrednosti je m, ker je polinom stopnje m. $(A - \lambda I) = 0 \rightarrow$ lastni vektor dobimo tako, da rešimo ta homogen sistem.

① Ali je $\begin{bmatrix}1\\-1\end{bmatrix}$ lastni vektor $A=\begin{bmatrix}1&-1\\0&2\end{bmatrix}$? Lastna vrednost?

$Av=\lambda v$

$Av-\lambda v=0$

$(A-\lambda I)v=0$

$\det(A-\lambda I)=0$

$v=\begin{bmatrix}1\\-1\end{bmatrix}$

$A=\begin{bmatrix}1&-1\\0&2\end{bmatrix}$

$$A\cdot v=\begin{bmatrix}1&-1\\0&2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2\\-2\end{bmatrix}=2\begin{bmatrix}1\\-1\end{bmatrix}=2v$$

v je lastni vektor matrike A, $\lambda=2$ je pripadajoča lastna vrednost.

Lastni vektorji, ki pripadajo različnim lastnim vrednostim so LINEARNO NEODVISNI. To potrebujemo za diagonalizacijo matrike.

DIAGONALIZACIJA MATRIKE

→ matriko se da diagonalizirati če ima vse lastne vrednosti $\mathbb{R}$ in med seboj različne

→ če je matrika A simetrična, se da diagonalizirati, ima $\mathbb{R}$ lastne vrednosti, poiskati se da takšne lastne vektorje, da so med sabo pravokotni $P^{-1}=P^T$ (ortogonalne matrike)

$A=\begin{bmatrix} & \\ & \end{bmatrix}$ $\qquad A=\begin{bmatrix}0&0&0\\0&1&0\\1&0&1\end{bmatrix}$

$$\left.\begin{aligned}A&=PDP^{-1}\\A^{20}&=PD^{20}P^{-1}\end{aligned}\right\}\ \boxed{A^m=PD^mP^{-1}}$$

1) $$\boxed{\det(A-\lambda I)}=\begin{vmatrix}a_{11}-\lambda&a_{12}&a_{13}\\a_{21}&a_{22}-\lambda&a_{23}\\a_{31}&a_{32}&a_{33}-\lambda\end{vmatrix}=\begin{vmatrix}0-\lambda&0&0\\0&1-\lambda&0\\1&0&1-\lambda\end{vmatrix}=-\lambda(1-\lambda)^2$$

$\lambda_1=0 \qquad \lambda_{2,3}=1$

2) POIŠČI LASTNE VEKTORJE ZA LASTNE VREDNOSTI

$\lambda_1=0$

$$\boxed{[A-I\,|\,\begin{smallmatrix}0\\0\\0\end{smallmatrix}]}=\left[\begin{array}{ccc|c}0&0&0&0\\0&1&0&0\\1&0&1&0\end{array}\right]\begin{array}{l}\to\emptyset\\\to \boxed{y=0}\\\to x+z=0\quad \boxed{z=\alpha}\ \alpha\in\mathbb{R}\\\quad\boxed{x=-\alpha}\end{array}$$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\alpha\begin{bmatrix}-1\\0\\1\end{bmatrix}$ $\qquad \boxed{v_1=\begin{bmatrix}-1\\0\\1\end{bmatrix}}$

$\lambda_{2,3}=1$ → rabiš 2 lastna vektorja!

$$[A-I\,|\,\begin{smallmatrix}0\\0\\0\end{smallmatrix}]=\left[\begin{array}{ccc|c}-1&0&0&0\\0&0&0&0\\1&0&0&0\end{array}\right]\to \begin{array}{l}\boxed{x=0}\\\boxed{y=\alpha}\\\boxed{z=\beta}\end{array}\ \alpha,\beta\in\mathbb{R}$$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\alpha\begin{bmatrix}0\\1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\0\\1\end{bmatrix}$ $\qquad \boxed{v_2=\begin{bmatrix}0\\1\\0\end{bmatrix}\quad v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}}$

3) DOBIŠ D in P

$D=\begin{bmatrix}\lambda_1&&0\\&\lambda_2&\\0&&\lambda_3\end{bmatrix}$ $\qquad P=\begin{bmatrix}v_1\,|\,v_2\,|\,v_3\end{bmatrix}$ $\to D=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $\qquad P=\begin{bmatrix}-1&0&0\\0&1&0\\1&0&1\end{bmatrix}$

$\lambda_1 \qquad \lambda_{2,3}$

4) 3 MOŽNOSTI $A^m=PD^mP^{-1}$

① $A=A^m$

$D=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$D^{20}=\begin{bmatrix}0^{20}&0&0\\0&1^{20}&0\\0&0&1^{20}\end{bmatrix}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$D=D^{20}$

$A=A^{20}$

② $A^m=I$

$A^{13}=PD^{13}P^{-1}=P\begin{bmatrix}1^{13}&0&0\\0&1^{13}&0\\0&0&(-1)^{13}\end{bmatrix}P^{-1}=$

$=P\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}P^{-1}=P\cdot I\,P^{-1}=I(PP^{-1})$

$\quad\ \ \, \| \qquad\qquad\qquad\qquad\ \ \|$
$\quad\ \ \, I \qquad\qquad\qquad\qquad\ \ I$

$A^{13}=I$

③ ni točka ① in ne točka ② → izračunaš

$PD^mP^{-1}=A$

1) narediš inverz $P=P^{-1}$

2) množiš PD^mP^{-1}

3) dobiš A

9. VRSTE

a_m = številsko zaporedje

Temu zaporedju priredimo zaporedje DELNIH VSOT s členi $S_k = \sum_{m=1}^{k} a_m$. Limito dobljenega zaporedja imenujemo ŠTEVILSKA VRSTA:

$$\boxed{\sum_{m=1}^{\infty} a_m := \lim_{k\to\infty} S_k}$$

1. Vrsta je KONVERGENTNA natanko tedaj, ko je konvergentno zaporedje $(S_k)_{k\in\mathbb{N}}$. Vsota vrste je enaka limiti zaporedja $(S_k)_{k\in\mathbb{N}}$, če ta obstaja. Sicer je vsota DIVERGENTNA.

$$\left.\begin{array}{l} S_1 = a_1 \\ S_2 = a_1 + a_2 \\ S_3 = a_1 + a_2 + a_3 \\ \quad \ldots \\ S_m = a_1 + \ldots + a_m \end{array}\right] \to \text{ZAPOREDJE DELNIH VSOT } (S_m)_{m\in\mathbb{N}}$$

---

* $\sum_{m=1}^{\infty} \underbrace{\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k}}}_{a_k}$

$S_1 = a_1 = \frac{1}{\sqrt{2}} - 1$

$S_2 = a_1 + a_2 = \cancel{\frac{1}{\sqrt{2}}} - 1 + \frac{1}{\sqrt{3}} - \cancel{\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{3}} - 1$

$S_3 = \underbrace{a_1 + a_2}_{S_2} + a_3 = \cancel{\frac{1}{\sqrt{3}}} - 1 + \frac{1}{\sqrt{4}} - \cancel{\frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{4}} - 1$

$S_4 = \frac{1}{\sqrt{5}} - 1$

...

$\hat{S}_m = \frac{1}{\sqrt{m+1}} - 1$

$(S_m = S_{m-1} + a_m)$

$S_m = \hat{S}_m \; (\forall m \in \mathbb{N}) \to$ INDUKCIJA

$m=1$ ✓

$m=k$ (iP)

$\underset{L}{S_k} = \underset{D}{\hat{S}_k} \left(= \frac{1}{\sqrt{k+1}} - 1\right)$

$\hat{S}_{k+1} = \frac{1}{\sqrt{k+2}} - 1$

$m = k+1$

$L = \underbrace{S_{k+1} = S_k + a_{k+1}}_{} \overset{iP}{=} \underbrace{\cancel{\frac{1}{\sqrt{k+1}}} - 1}_{} + \underbrace{\frac{1}{\sqrt{k+2}} - \cancel{\frac{1}{\sqrt{k+1}}}}_{a_{k+1}} = \frac{1}{\sqrt{k+2}} - 1 = \hat{S}_{k+1} = D$ ✓

$S_k = \overset{\to 0}{\frac{1}{\sqrt{k+1}}} - 1 \xrightarrow{k\to\infty} -1$ = VRSTA JE KONVERGENTNA (obstaja limita) ~ NJENA VSOTA JE -1

* $\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right)$ — a_k

$S_1 = a_1 = \ln(1+1) = \ln 2$

$S_2 = a_1 + a_2 = \ln\left(1+\frac{1}{2}\right) + \ln 2 = \ln\left(\frac{3}{2}\cdot 2\right) = \ln 3$

$S_3 = a_1 + a_2 + a_3 = \ln 3 + \ln\left(\frac{4}{3}\right) = \ln\left(3\cdot\frac{4}{3}\right) = \ln 4$

...

$\hat{S}_m = \ln(m+1)$

$(S_m = \hat{S}_m \quad (= \ln(m+1)))$

DOKAZ INDUKCIJA:

$m=1$ ✓

$m=k$ (iP)

$S_k = \hat{S}_k \quad (= \ln(m+1))$

$m = k+1$

$\hat{S}_{k+1} = \ln(k+1+1) = \ln(k+2)$

$L = S_k + a_{k+1} \overset{iP}{=} \ln(k+1) + \ln\left(1+\frac{1}{k+1}\right) = \ln\left(\cancel{(k+1)}\cdot\left(\frac{k+1+1}{\cancel{k+1}}\right)\right) = \ln(k+2) = \hat{S}_{k+1} = D$

$\lim_{m\to\infty} S_m = \lim_{m\to\infty} \ln(m+1) = \infty \to$ VRSTA DIVERGIRA (limita ne obstaja)

(1.) GEOMETRIJSKA VRSTA $\boxed{\sum_{m=1}^{\infty} a q^{m-1}}$ je KONVERGENTNA za $|q|<1$ in v tem primeru je njena vsota $\frac{a}{1-q}$

$$\boxed{\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}\ ; \quad |q|<1}$$

* $\sum_{m=1}^{\infty} \frac{3}{5^m} = \sum_{m=1}^{\infty} 3\cdot\left(\frac{1}{5}\right)^m = 3\cdot\frac{1}{5} + 3\cdot\left(\frac{1}{5}\right)^2 + 3\cdot\left(\frac{1}{5}\right)^3 + \ldots = \frac{1}{5}\sum_{m=1}^{\infty} 3\cdot\left(\frac{1}{5}\right)^{m-1} = \frac{1}{5}\cdot 3\cdot\frac{1}{1-\frac{1}{5}} = \frac{3}{4}$ — q

* $\sum_{m=0}^{\infty} \frac{3}{5^m} = \sum_{m=0}^{\infty} 3\cdot\left(\frac{1}{5}\right)^m = \sum_{m=1}^{\infty} 3\cdot\left(\frac{1}{5}\right)^{m-1} = 3\cdot\frac{1}{1-\frac{1}{5}} = \frac{15}{4}$ — q

* $\sum_{m=0}^{\infty} \left(\frac{3}{5}\right)^m = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$

* $\sum_{m=0}^{\infty} \left(\frac{5}{3}\right)^m = \frac{1}{1-\left(\frac{5}{3}\right)} =$ — q

$q = \frac{5}{3} > 1 \to$ DIVERGENTNA VRSTA

POGOJ ZA KONVERGENTNO VRSTO: $\boxed{\lim_{m\to\infty} |a_m| = 0}$

* $\sum_{m=1}^{\infty} \frac{3^m}{m}$ — a_m

$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{3^m}{m} = \frac{\frac{3^m}{3^m} \to 1}{\frac{m}{3^m} \to 0} = \infty$ NE!

* $\sum_{m=1}^{\infty} \frac{1}{m3^m}$

$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \frac{\frac{1}{3^m} \to 0}{\frac{m3^m}{3^m}} = \lim_{m\to\infty} \frac{0}{m} = 0$ DA!

* $\sum_{m=1}^{\infty} \left(\frac{7}{5}\right)^m$

$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \left(\frac{7}{5}\right)^m = \infty$

$\to \frac{7}{5} > 0 = \infty$ NE!

* $\sum_{m=1}^{\infty} \left(-\frac{3}{4}\right)^m$

$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \left(-\frac{3}{4}\right)^m = 0$ DA!

$\to \lim_{m\to\infty} |a_m| = 0$

$\lim_{m\to\infty} \left|\left(-\frac{3}{4}\right)^m\right| = 0$

$\lim \left(\frac{3}{4}\right)^m = 0$

$\to \frac{3}{4} < 0 = \varnothing$

* $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+3}{n^2}\right)$ $\qquad \lim_{n\to\infty} \underbrace{\left|(-1)^{n+1}\left(\frac{n+3}{n^2}\right)\right|}_{\text{VEDNO}\oplus} = \lim_{n\to\infty} \left(1 \cdot \frac{\frac{n}{n^2}+\frac{3}{n^2}}{\frac{n^2}{n^2}}\right) = 0$ DA!

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}$

(2) Vrsta s $\oplus$ členi: $\sum_{n=1}^{\infty} a_n$, kjer so vsi $a_n \geq 0$, je konvergentna natanko tedaj, ko je zaporedje njenih delnih vsot omejeno.

PRIMERJALNI KRITERIJ (velja $0 \leq a_n \leq b_n$ za $\forall n \in \mathbb{N}$)

1) če je vrsta $\sum_{n=1}^{\infty} b_n$ konvergentna, je konvergentna tudi vrsta $\sum_{n=1}^{\infty} a_n$

2) če je vrsta $\sum_{n=1}^{\infty} a_n$ divergentna, je divergentna tudi vrsta $\sum_{n=1}^{\infty} b_n$

(3) HARMONIČNA VRSTA $\boxed{\sum_{n=1}^{\infty}\frac{1}{n}}$ je DIVERGENTNA!

$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots = \infty$

(tudi če $\lim_{n\to\infty}\frac{1}{n}=0$)

* $\sum_{n=1}^{\infty} \frac{|\sin n|}{5^n}$

$a_n = \frac{|\sin n|}{5^n} \leq \frac{1}{5^n} = \left(\frac{1}{5}\right)^n = b_n$ $\qquad |\sin n| \leq 1$ $\qquad$ GEOMETRIJSKO ZAPOREDJE $q=\frac{1}{5}$

$|q|<1 \to \sum_{n=1}^{\infty} b_n$ KONVERGENTNO

* $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n}}$

$a_n = \frac{1}{\sqrt[4]{n}} \geq \frac{1}{n} = b_n$ $\qquad$ b_n je HARMONIČNO ZAPOREDJE $\to \sum_{n=1}^{\infty} b_n \to$ DIVERGENTNO

* $\sum_{n=1}^{\infty}\frac{3^n(\sqrt{n})}{2^n}$

$a_n = \frac{3^n}{2^n}\sqrt{n} \geq \left(\frac{3}{2}\right)^n = b_n$ $\qquad \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ $\qquad$ GEOMETRIJSKO ZAPOREDJE $q=\frac{3}{2}>1$ $\qquad$ DIVERGENTNO $\sum_{n=1}^{\infty} b_n$

DIVERGENTNO $\sum_{n=1}^{\infty} a_n$

* $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$

$a_n = \frac{1}{\sqrt{n(n+1)}} < \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ $\qquad n < (n+1)$

$a_n = \frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+1)}} = \frac{1}{n+1}$ $\qquad$ HARMONIČNO ZAPOREDJE (brez 1. člena) $\to$ DIVERGENTNO

$\Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGENTNO

(4) ALTERNIRAJOČA VSOTA $\boxed{\sum_{n=1}^{\infty}(-1)^n a_n}$, kjer je $a_n \geq 0$

LEIBNIZOV KRITERIJ: če je zaporedje $(a_n)_{n\in\mathbb{N}}$ ① MONOTONO in ② $\lim_{n\to\infty} a_n = 0$ potem je alternirajoča vrsta $\sum_{n=1}^{\infty}(-1)^n a_n$ KONVERGENTNA. V tem primeru velja OCENA ZA NAPAKO

$(-1)^n \to (-1)^{n+1}$, $(-1)^{n-1}$

$\boxed{|S-S_n| < a_{n+1}}$, kjer je S vsota vrste in S_n delna vsota vrste.

$S \to$ vsota vrste

$S_n \to$ delna vsota vrste

$|S-S_n| = R_n \to$ napaka

(5) **ABSOLUTNO KONVERGENTNA VRSTA** $\boxed{\sum_{n=1}^{\infty} a_n}$ je absolutno konvergentna. Vsaka absolutno konvergentna vrsta je konvergentna

KORENSKI TEST: če je $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ je vrsta absolutno konvergentna, če je ta limita > 1 je divergentna

$\boxed{q_n = \sqrt[n]{|a_n|}}$

KVOCIENTNI TEST: če je $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ je vrsta $\sum_{n=1}^{\infty} a_n$ absolutno konvergentna, če je ta limita > 1 pa divergentna.

$\boxed{q_n = \left|\frac{a_{n+1}}{a_n}\right|}$

* $\sum_{n=1}^{\infty} \frac{n^3}{n!}$

$a_n = \frac{n^3}{n!}$ KVOCIENTNI TEST: $q_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^3}{(n+1)!}}{\frac{n^3}{n!}}\right| =$

$$= \frac{n^3 \cdot (n+1)!}{(n+1)^3 \; n!} = \frac{n^3 \cdot \cancel{n!}\,\cancel{(n+1)}}{(n+1)^{\cancel{3}\,2} \cdot \cancel{n!}}$$

$$= \frac{(n+1)^2}{n^3} \xrightarrow{n\to\infty} 0 < 1$$

VRSTA KONVERGIRA ABSOLUTNO

$$\boxed{\begin{aligned} n! &= 1\cdot 2\cdot 3\cdot \ldots \cdot n \\ (n+1)! &= \underbrace{1\cdot 2\cdot 3\cdot \ldots \cdot n}_{n!} \cdot (n+1) \\ &= n!\,(n+1) \end{aligned}}$$

* $\sum_{n=1}^{\infty} \frac{2^n}{n}$

$a_n = \frac{2^n}{n}$

PRIMERJALNI KRITERIJ

$a_n = \frac{2^n}{n} > \frac{1}{n} = b_n$ HARMONIČNO ZAPOREDJE

$\Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGIRA

KORENSKI TEST

$$q_n = \sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n}{n}} = \frac{\sqrt[n]{2^n}}{\sqrt[n]{n}} = \frac{2}{\sqrt[n]{n}} \xrightarrow{n\to\infty} 2 > 1$$

$\sqrt[n]{n} \to 1$

$\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n}{n}$ DIVERGIRA

KVOCIENTNI TEST

$$q_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \frac{\cancel{2^n}\cdot 2\cdot n}{\cancel{2^n}(n+1)} = \frac{2n}{n+1} \xrightarrow{n\to\infty} \frac{2}{1} = 2 > 1$$

$2n \to 2$, $n \to 1$, $1 \to 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGIRA

* $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n^2}\right)$ a_n

ALTERNIRAJOČE ZAPOREDJE $\left(\frac{n+1}{n^2}\right)$

LEIBNIZOVO PRAVILO

1) $a_n \to$ padajoče

$a_n > a_{n+1}$

$\frac{n+1}{n^2} > \frac{n+2}{(n+1)^2} \;/\; (n+1)^2\, n^2$ $\;\nearrow 0 \quad \nearrow 0$

$(n+1)^3 > n^2(n+2)$

$\cancel{n^3} + 3n^2 + 3n + 1 > \cancel{n^3} + 2n^2$

$3n^2 + 3n + 1 > 2n^2$ ✓

$\Rightarrow a_n$ povsod pada $\to \boxed{\text{MONOTONO}}$ ✓

2) $\boxed{\lim_{n\to\infty} a_n = 0}$

$\lim_{n\to\infty} \frac{n+1}{n^2} = 0$ ✓

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n^2}\right)$ KONVERGIRA

Ali je tudi absolutno konvergentno? $\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2}\right)$

KVOCIENTNI KRITERIJ $q_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{n+2}{(n+1)^2}}{\frac{n+1}{n^2}}\right| = \frac{(n+2)n^2}{(n+1)^3} \xrightarrow{n\to\infty} 1$ ODPADE

PRIMERJALNI KRITERIJ $a_n = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} > \frac{1}{n} = b_n$ HARMONIČNO ZAPOREDJE $\Rightarrow$ DIVERGIRA

$\Rightarrow$ tudi $\sum_{n=1}^{\infty} a_n$ DIVERGIRA $\rightarrow \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{n+1}{n^2}\right)$ je POGOJNO KONVERGENTNO $\rightarrow$ TORej NI ABSOLUTNO KONVERGENTNO

$$\boxed{\left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}} \xrightarrow{n\to\infty} \frac{1}{e}}$$

* koliko členov ~~zaporedja~~ vrste $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ moramo seštéti, da bo napaka gotovo manjša od 0,05?

$R_n = |S - S_n| < a_{n+1} < 0{,}05$

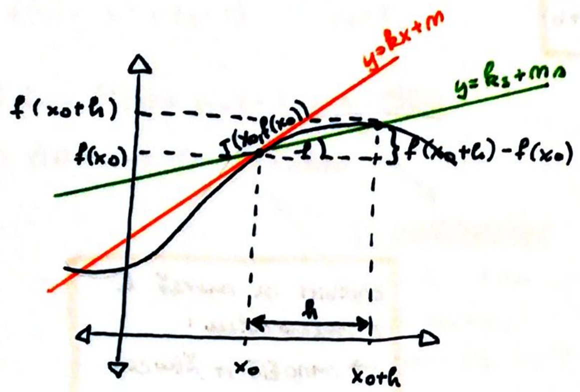
$\frac{1}{\sqrt{n+1}} < \frac{5}{100} \quad / \cdot 20 \cdot \sqrt{n+1}$

$20 < \sqrt{n+1} \quad /^2$

$400 < n+1$

$399 < n$ $\rightarrow$ seštéti moramo prvih 399 členov.

11. ODVODI



$$k_s = tg\,\varphi = \frac{f(x_0+h)-f(x_0)}{h}$$

$$k_T = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

PRAVILA

- Če sta funkciji f in g odvedljivi v točki x_0, so v točki x_0 odvedljive tudi $f+g$, $f\cdot g$, λf $(\lambda\in\mathbb{R})$ in $\frac{f}{g}$ $(g(x)\neq 0)$ ker velja:
  - $(f+g)'(x_0) = f'(x_0)+g'(x_0)$
  - $(f\cdot g)'(x_0) = f'(x_0)g(x) + f(x_0)g'(x_0)$
  - $(\lambda f)'(x_0) = \lambda\cdot f'(x_0)$
  - $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g^2(x_0)}$

- VERIŽNO PRAVILO ALI ODVOD KOMPOZITUMA

  Če je funkcija f odvedljiva v točki x_0 in je funkcija g odvedljiva v točki $f(x_0)$, je kompozitum $g\circ f$ odvedljiva v x_0 in velja
  - $(g\circ f)'(x_0) = g'(f(x))\cdot f'(x_0)$

- ODVODI NEKATERIH ELEMENTARNIH FUNKCIJ
  - $\lambda' = 0$ $(\lambda\in\mathbb{R})$
  - $(x^\alpha)' = \alpha x^{\alpha-1}$ $(\alpha\in\mathbb{R})$
  - $(\ln x)' = \frac{1}{x}$
  - $(e^x)' = e^x$
  - $(\sin x)' = \cos x$
  - $(\cos x)' = -\sin x$
  - $(tg\,x)' = \frac{1}{\cos^2 x}$
  - $(\cot x)' = -\frac{1}{\sin^2 x}$
  - $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
  - $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
  - $(\mathrm{arctg}\,x)' = \frac{1}{1+x^2}$

- SESTAVLJENE FUNKCIJE
  - $f(x) = x\sqrt{1-2x} = x(1-2x)^{\frac{1}{2}} \Rightarrow f'(x) = \overbrace{1\cdot(1-2x)^{\frac{1}{2}} + x\cdot\underbrace{\tfrac{1}{2}(1-2x)^{-\frac{1}{2}}\cdot(-2)}_{\text{odvod sestavljene funkcije}}}^{\text{MNOŽENJE}}$
  - $f(x) = e^{\sin 3x} \Rightarrow f'(x) = e^{\sin 3x}\cdot(\sin 3x)' = e^{\sin 3x}\cdot\cos(3x)\cdot(3x)' = e^{\sin(3x)}\cdot\cos(3x)\cdot 3$

- da bo funkcija ODVEDLJIVA (mora biti ZVEZNA)

$$f(x)=\begin{cases} x^2 ; & x \geq 1 \\ ax+b ; & x<1 \end{cases}$$

$$\boxed{\lim_{x \uparrow 1} x^2 = 1 = \lim_{x \downarrow 1}(ax+b)}$$

$1 = a+b \rightarrow a+b-1=0$

ODVOD V 1

$$\left(\lim_{x\uparrow 1} f'(x) = f'(1) = \lim_{x \downarrow 1} f(x)\right)$$

$$\underbrace{\lim_{x\uparrow 1} f(x) = \lim_{x \uparrow 1}(ax+b) = a+b \qquad \cancel{\lim} \; f(1)=1}$$

$a+b=1 \rightarrow a+b-1=0$

zmebiti se moraš h v imenovalcu:
- → množiš s števcem
- → izpostaviš h v števcu

$$\lim_{h\uparrow 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\uparrow 0}\frac{a(1+h)+b-1}{h} = \lim_{h\uparrow 0}\frac{a+ha+b-1}{h} = \lim_{h\uparrow 0}\left(\frac{ha}{\cancel{h}}+\frac{a+b-1}{h}\right)=a$$

$$\lim_{h\downarrow 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\downarrow 0}\frac{(1+h)^2-1}{h} = \lim_{h\downarrow 0}\frac{1+2h+h^2-1}{h} = \lim_{h\downarrow 0}\frac{\cancel{h}(2+h)}{\cancel{h}} = \lim_{h\to 0}(2+h)=2$$

$\rightarrow a=2$

$a+b=1 \rightarrow 2+b=1 \rightarrow b=-1$

- $$f(x)=\begin{cases} \sqrt{x^3} ; & x \geq 1 \\ ax^2+bx ; & x<1 \end{cases}$$

$x=1$?
$1=ax^2+bx$
$1=a+b$

$$\lim_{x\uparrow 1} f(x) = \lim_{x\uparrow 1}\sqrt{x^3} = 1 = \lim_{x\downarrow 1}(ax^2+bx)$$

ODVOD V 1

$$\lim_{h\uparrow 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\uparrow 0}\frac{a(1+h)^2+b(1+h)-1}{h} = \lim_{h\uparrow 0}\frac{a+2ha+h^2a+bh-1+b}{h} = \lim_{h\uparrow 0}\frac{\cancel{1}+2ha+h^2a+bh-\cancel{1}}{h} =$$

deliš s števcem

$$= \lim_{h\uparrow 0}\frac{h(2a+ha+b)}{\cancel{h}} = \lim_{h\uparrow 0}(2a+ha+b) = 2a+b$$

množiš z imenovalcem

$$\lim_{h\downarrow 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\downarrow 0}\frac{\sqrt{(1+h)^3}-1}{h} = \lim_{h\downarrow 0}\frac{(1+h)\sqrt{(1+h)}-1}{h} = \lim_{h\downarrow 0}\frac{(1+h)^3-1}{h\left(\sqrt{(1+h)^3}+1\right)} =$$

$$= \lim_{h\downarrow 0}\frac{(1+h)(1+h)(1+h)-1}{h\left(\sqrt{(1+h)^3}+1\right)} = \lim_{h\downarrow 0}\frac{\cancel{1}+h+2h+2h^2+h^2+h^3-\cancel{1}}{h\left(\sqrt{(1+h)^3}+1\right)} = \lim_{h\downarrow 0}\frac{\cancel{h}(h^2+3h+3)}{\cancel{h}\left(\sqrt{(1+h)^3}+1\right)} = \frac{0+0+3}{1+1} = \frac{3}{2}$$

$2a+b=\frac{3}{2}$

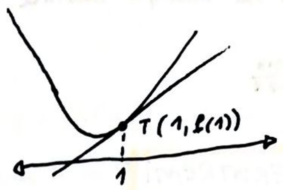
$a+b=1 \rightarrow a=1-b$

$2(1-b)+b=\frac{3}{2} \rightarrow b=\frac{1}{2}$

$a=\frac{1}{2}$

ENAČBA TANGENTE NA KRIVULJO V DOLOČENI TOČKI

$f(x) = x^2 + 2x - 12 \qquad x_0 = 1$



① $k_T = f'(1) = 2x+2 = 2+2 = 4 = f'(x_0)$

② $f(1) = 1+2-12 = -9 = f(x_0)$

③ $T(1,-9)$

④ $y = k_T x + m_T$

$y = 4x + m_T$

$-9 = 4x + m_T$

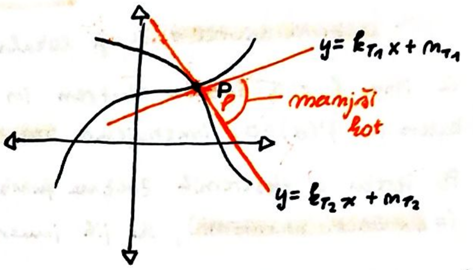
$m_T = -13 \longrightarrow y = 4x - 13$

KOT MED KRIVULJAMA

$f(x) = \frac{x+1}{x+2}$

$g(x) = \frac{x^2+4x+8}{16}$

$\boxed{tg\,\varphi = \left|\frac{k_1 - k_2}{1+k_1k_2}\right|}$ → $p_1 \parallel p_2 \to k_1 = k_2$; $p_1 \perp p_2 \to k_1 = -\frac{1}{k_2}$



① PRESEČIŠČE → da dobiš določene točke

$\frac{x+1}{x+2} = \frac{x^2+4x+8}{16}$

$(x+1)\cdot 16 = (x^2+4x+8)(x+2)$

$16x + 16 = x^3 + 2x^2 + 4x^2 + 8x + 8x + 16$

$x^2(x+6) = 0$

$x_{1,2} = 0 \to y = \frac{1}{2} \qquad P_1(0, \frac{1}{2})$

$x_3 = -6 \to y = \frac{5}{4} \qquad P_2(-6, \frac{5}{4})$

*tangenta mora pokrati skozi izhodišče $(0,0) \to m_T = 0$
→ enačba tangente:
$y = k_T x$

② ENAČBI TANGENT NA KRIVULJI v določenih točkah (presečiščih)

$P_1(0, \frac{1}{2})$

1) $f'(0) = \frac{x+2-x-1}{(x+2)^2} = \frac{1}{4} = k_{T_1} \qquad f(0) = \frac{1}{2} = m_{T_1}$

2) $g'(0) = \frac{(2x+4)16}{16^2} = \frac{1}{4} = k_{T_2} \qquad g(0) = \frac{1}{2} = m_{T_2}$

$\to tg\,\varphi = \left|\frac{\frac{1}{4} - \frac{1}{4}}{1+\frac{1}{4}\cdot\frac{1}{4}}\right| \Rightarrow \varphi = 0°$

$P_2(-6, \frac{5}{4})$

1) $f'(-6) = \frac{1}{16} = k_{T_1} \qquad f(-6) = \frac{13}{8} = m_{T_1}$

2) $g'(-6) = -\frac{1}{2} = k_{T_2} \qquad g(-6) = -\frac{7}{4} = m_{T_2}$

$\to tg\,\varphi = \left|\frac{\frac{1}{16} + \frac{1}{2}}{1+\frac{1}{16}\cdot\frac{1}{2}}\right| \Rightarrow \varphi = arctg\,\frac{18}{31}$

L'HÔPITALOV IZREK

naj bosta f in g odvedljivi funkciji na (a,b) in za točko $\alpha \in (a,b)$, velja $f(\alpha) = g(\alpha) = 0$, ter $g(x) \neq 0$ za $x \neq \alpha$, če obstaja limita: $\boxed{L = \lim_{x\to\alpha} \frac{f'(x)}{g'(x)}}$

potem obstaja tudi $\boxed{\lim_{x\to\alpha} \frac{f(x)}{g(x)} = L}$. Izrek velja tudi za limite oblike $\lim_{x\to\alpha} \frac{f(x)}{g(x)}$, ko je $f(\alpha) = g(\alpha) = \infty$,
, ko je $a = \pm\infty$,
, ter za enostranske limite

- $\lim_{x\to4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} \overset{L'H}{=} \lim_{x\to4} \frac{\frac{1}{2}(1+2x)^{-\frac{1}{2}}\cdot 2}{\frac{1}{2}(x)^{-\frac{1}{2}}} = \lim_{x\to4} \frac{2\sqrt{x}}{\sqrt{1+2x}} = \frac{2\cdot2}{3} = \frac{4}{3}$

  ODVAJAŠ
  ODVAJAŠ

- $\lim_{x\to0} \frac{e^{x^2}-1}{1-\cos x} \overset{①\,L'H}{=} \lim_{x\to0} \frac{e^{x^2}(x^2)'}{\sin x} = \lim_{x\to0} \frac{2e^{x^2}\cdot x}{\sin x} \overset{②\,L'H}{=} \lim_{x\to0} \frac{2e^{x^2}\cdot 2x\cdot x + 2e^{x^2}\cdot 1}{\cos x} = \lim_{x\to0} \frac{4x^2\cdot e^{x^2} + 2e^{x^2}}{\cos x} = \frac{0+2}{1} = 2$

- $\lim_{x\downarrow0} (x\cdot \ln x) = \lim_{x\downarrow0} \frac{\ln x}{\frac{1}{x}} \overset{L'H}{=} \lim_{x\downarrow0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\downarrow0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\downarrow0} \frac{x^{\cancel{2}}}{\cancel{x}} = -\lim_{x\downarrow0} x = 0$

  narediš ulomek

* ne odvajaš tako kot se odvaja ulomke ampak po L'Hopitalu (odvajaš števec in ločeno odvajaš imenovalec)
* $\operatorname{tg} x$ spremeniš v $\frac{\sin x}{\cos x}$

STACIONARNE TOČKE, EKSTREMI

naj bo $f: (a,b) \to \mathbb{R}$. Točka $\alpha \in (a,b)$ se imenuje:

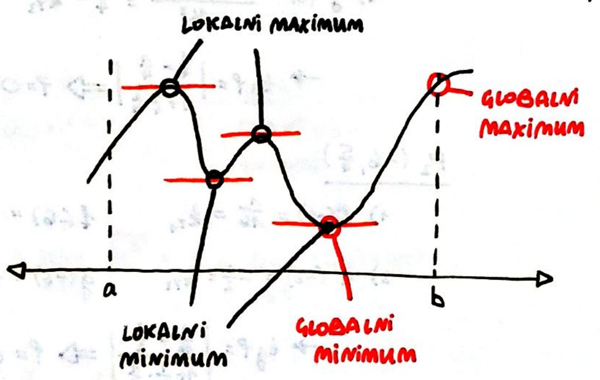
- **LOKALNI MAKSIMUM** funkcije f, če obstaja ε-okolica $\mathcal{U}_\varepsilon(\alpha) := (\alpha - \varepsilon, \alpha + \varepsilon)$ točke α, da velja:
  $\boxed{f(x) \leq f(\alpha)}$ za vse $x \in \mathcal{U}_\varepsilon(\alpha)$
- **LOKALNI MINIMUM** funkcije f, če obstaja ε-okolica točke α, da velja:
  $\boxed{f(x) \geq f(\alpha)}$ za vse $x \in \mathcal{U}_\varepsilon(\alpha)$
- **LOKALNI EKSTREM**, če je lokalni maksimum ali lokalni minimum

če ima f v α lokalni ekstrem in je f odvedljiva v α, potem je $\boxed{f'(\alpha) = 0}$. Točka α, za katero je $f'(\alpha) = 0$, imenujemo **STACIONARNA TOČKA** funkcije f.

Po izreku o ekstremih zvezna funkcija $f: [a,b] \to \mathbb{R}$ na $[a,b]$ doseže ekstremne vrednosti (= **GLOBALNI EKSTREMI**), ki jih imenujemo:

- $x_M \in [a,b]$ je **GLOBALNI MAXIMUM** funkcije f, če $\boxed{f(x) \leq f(x_M)}$ za vse $x \in [a,b]$
- $x_m \in [a,b]$ je **GLOBALNI MINIMUM** funkcije f, če $\boxed{f(x) \geq f(x_m)}$ za vse $x \in [a,b]$

če je funkcija f zvezna na $[a,b]$ in odvedljiva na (a,b), potem doseže svoje ekstremne vrednosti bodisi v STACIONARNIH TOČKAH, bodisi v KRAJIŠČIH INTERVALA.



- $f(x) = e^{-2x} \sin^2 x$ na $[0, 2\pi]$

① ↳ **zvezna** na $\mathbb{R}$ ($\to$ potem tudi na $[0,2\pi]$) ✓
↳ **odvedljiva** na $\mathbb{R}$ ($\to$ potem tudi na $[0,2\pi]$) ✓

② **STACIONARNE TOČKE $f'(x) = 0$**

1) ODVOD
$$f'(x) = -2e^{-2x} \cdot \sin^2 x + 2e^{-2x} \cdot \sin x \cdot \cos x$$

2) $f'(x) = 0$
$$0 = -2e^{-2x} \cdot \sin^2 x + 2e^{-2x} \cdot \sin x \cdot \cos x$$
$$0 = 2 e^{-2x} \sin x \underbrace{(-\sin x + \cos x)}$$

e^{-2x}: $\neq 0$

$\sin x$: $0° + k\pi$ (na intervalu $[0,2\pi]$)
$x_1 = 0$
$x_2 = \pi$
$x_3 = 2\pi$

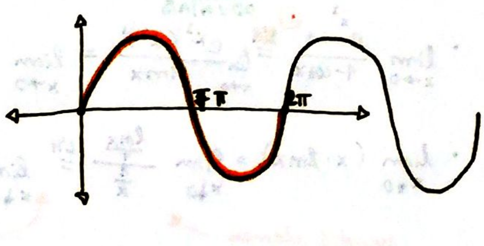
$\sin x = \cos x$
$x_4 = \frac{\pi}{4}$
$x_5 = \frac{5\pi}{4}$

③ **VREDNOSTI KANDIDATOV** ZA EKSTREME
vstaviš v originalno funkcijo

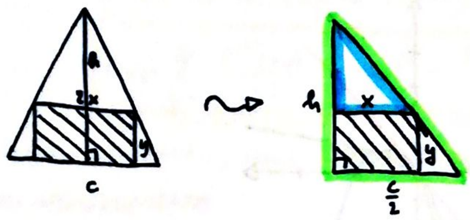
krajišči:
$a = x_1 = 0 \to f(0) = e^{-2\cdot 0} \cdot \sin^2 0 = 0$
$b = x_3 = 2\pi \to f(2\pi) = e^{-2\cdot 2\pi} \cdot \sin^2 2\pi = 0$ $\to$ MIN
$x_2 = \pi \to f(\pi) = 0$

$x_4 = \frac{\pi}{4} \to f(\frac{\pi}{4}) = \frac{1}{2e^{\frac{\pi}{2}}} \to$ MAX

$x_5 = \frac{5\pi}{4} \to f(\frac{5\pi}{4}) = \frac{1}{2e^{\frac{5\pi}{2}}}$



• ENAKOKRAKI TRIKOTNIK z osnovnico c in višino h ima VČRTAN PRAVOKOTNIK z osnovnico na stranici c, tako da bo PLOŠČINA NAJVEČJA!



$pl_{\square} = x \cdot y = pl(x,y)$

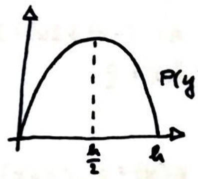
trikotnika ▲ in Δ:

$\frac{x}{h-y} = \frac{\frac{c}{2}}{h} \rightarrow x = \frac{c(h-y)}{2h}$

KVADRATNA funkcija

$pl\left(\frac{c(h-y)}{2h}, y\right) = \frac{c(h-y)}{2h} \cdot y = \frac{c}{2h}(h-y)y = \frac{c}{2h}(hy-y^2) = P(y)$

iščemo max za $P(y)$ na $[0,h] \Rightarrow P(y)$ doseže max v točki $\frac{h}{2} = y$

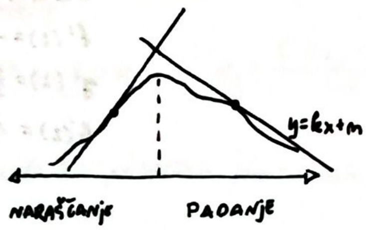


$x = \frac{c(h-\frac{h}{2})}{2h} = \frac{c\frac{h}{2}}{2h} = \frac{c}{4}$ (to je samo za $\frac{1}{2}$ trikotnika)

→ VČRTAN PRAVOKOTNIK ima DIMENZIJE $\frac{c}{2}, \frac{h}{2}$

• NARAŠČANJE, PADANJE

↓ $f'>0$ ↘ $f'<0$



$f(x) = \ln\left(\frac{2-x}{x^2}\right)$

1. $f'(x) = \frac{1}{\frac{2-x}{x^2}} \cdot \left(\frac{2-x}{x^2}\right)' = \frac{-x^2-(2-x)2x}{(2-x)x^2} = \frac{x-4}{(2-x)x}$

2. KDAJ $\frac{x-4}{(2-x)x}$ NARAŠČA / PADA? nariši graf

$y = \frac{x-4}{(2-x)x}$

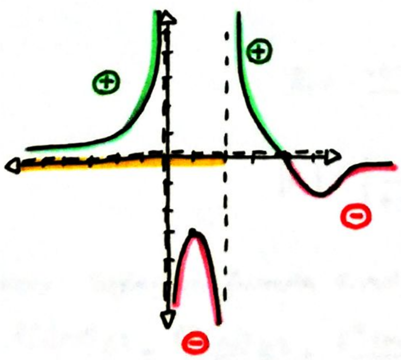
N: $x=4$

P: $x=2 \quad x=0$

Z: /

A: $(x-4):(2x-x^2)=0$

$f(1) = -3$



3. $f'>0$ (f narašča) $\Rightarrow x \in (-\infty,0) \cup (2,4)$

$f'<0$ (f pada) $\Rightarrow x \in (0,2) \cup (4,\infty)$

4. Definicijsko območje

$\frac{2-x}{x^2} > 0 \Leftrightarrow 2-x>0 \Rightarrow 2>x$

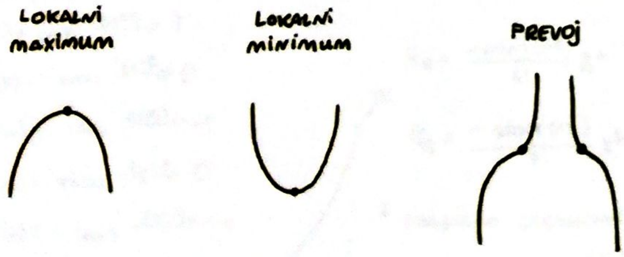
$2>x$

REŠITEV:

f - narašča $\rightarrow x \in (-\infty,0)$

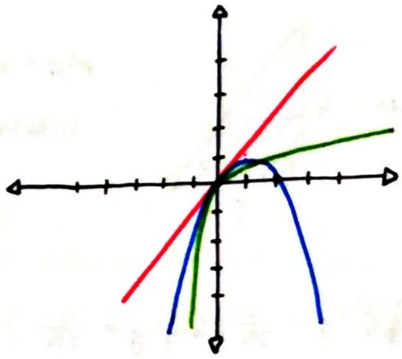
f - pada $\rightarrow x \in (0,2)$

• STACIONARNE TOČKE



• dokaži da velja za vsak $x > 0$

$$\underbrace{x - \frac{x^2}{2}}_{f(x)} < \underbrace{\ln(1+x)}_{g(x)} < \underbrace{x}_{h(x)}$$



① NARIŠI

$f(x) = x - \frac{x^2}{2} = x - \frac{1}{2}x^2$

N: $x_1 = 0$ $x_2 = 2$

P: /

Z_v: 0

A: $(-x^2 + 2x) : 2 = -\frac{1}{2}x^2 + x = y$

$f(1) = \frac{1}{2}$

$h(x) = x$

$g(x) = \ln(1+x)$

D_f: $1 + x > 0$, $-1 > x$ → $D_f = (-1, \infty)$

② PRESEČIŠČA $(0,0)$

→ ODVODI

$f'(x) = -\frac{1}{2}2x + 1 = 1 - x$ → $f'(0) = 1$

$g'(x) = \frac{1}{1+x} \cdot 1 = \frac{1}{1+x}$ → $g'(0) = 1$

$h'(x) = 1$ → $h'(0) = 1$

$f'(2) = -1$

$g'(2) = \frac{1}{3}$ $\qquad f(x) < g(x) < h(x)$

$h'(2) = 1$

DIFERENCIAL

TAYLORJEV IZREK

naj bo f $(n+1)$-krat zvezno odvedljiva funkcija na intervalu (a,b) in $x_0 \in (a,b)$. Potem za vse $x \in (a,b)$ velja:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{(n+1)}}$$ za nek $\xi \in (x_0, x)$

to je Taylorjeva formula, izraz $T_n(x)$ TAYLORJEV POLINOM funkcije f v točki x_0 stopnje n, ter $R_n(x)$ OSTANEK ali NAPAKA.

Druga oblika zapisa Taylorjeve formule ($x = x_0 + h$ in $\xi = x_0 + \theta h$ za nek $\theta \in (0,1)$)

$$\boxed{f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}h^k + \frac{f^{(n+1)}(x_0+\theta h)}{(n+1)!}h^{n+1}}$$

$$f(x_0+h) = f(x_0) + \frac{f'(x)}{1!}\cdot h + \frac{f''(x_0)}{2!}h^2 + \frac{f'''(x_0)}{3!} + \dots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n$$

$$R_n = \frac{f^{(n+1)}(x_0+\theta h)}{(n+1)!}h^{(n+1)} \quad ; \quad \theta \in [0,1]$$

Diferencial Taylorjeve formule REDA 1

$$f(x_0+h) = f(x_0) + f'(x_0)\cdot h + R_1 \qquad R_1 = \frac{f''(x_0+\theta h)}{2}h^2$$

- S pomočjo Taylorjeve formule reda 2 približno izračunaj in oceni napako

$$\frac{1}{1{,}03^2} = (1 + 3\cdot 10^{-2})^{-2} = 1 + \frac{-2}{1}\cdot(3\cdot 10^{-2}) + \frac{6}{2}(3\cdot 10^{-2})^2 + R_2 \approx 0{,}9427$$

$$f(x_0+h) = f(x_0) + \frac{f'(x_0)}{1}h + \frac{f''(x_0)}{2}h^2 + R_2$$

$$\begin{cases} f(x) = x^{-2} \xrightarrow{v\ x_0=1} 1 \\ f'(x) = -2x^{-3} \xrightarrow{v\ x_0=1} -2 \\ f''(x) = 6x^{-4} \xrightarrow{v\ x_0=1} 6 \end{cases}$$

$$f'''(x) = -24x^{-5}$$

$$R_2 = \frac{f'''(x_0+\theta h)}{3!}h^3$$

$$R_2 = \frac{-24(x_0+\theta h)^{-5}}{3\cdot 2\cdot 1}\cdot h^3$$

$$R_2 = \frac{-4(1+\theta\cdot 3\cdot 10^{-2})^{-5}}{1}\cdot(3\cdot 10^{-2})^3$$

$$|R_2| = \left|\frac{(3\cdot 10^{-2})^3}{4(1+\theta\cdot 3\cdot 10^{-2})^5}\right| \leq 4\cdot 3^3\cdot 10^{-6} = 108\cdot 10^{-6} = 0{,}000108$$

- Aproksimiraj funkcijo $f(x) = \sin x$. Taylorjeva formula 5. red, $x_0 = 0$

$$f(x_0+h) = f(x) + \frac{f'(x_0)}{1}h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \frac{f^{IV}(x_0)}{4!}h^4 + \frac{f^{V}(x_0)}{5!}h^5 + R_5$$

$$f(h) = 0 + \frac{1}{1}h + \frac{0}{2}h^2 + \frac{-1}{6}h^3 + \frac{0}{4!}h^4 + \frac{-1}{5!}h^5 + R_5 \approx h - \frac{1}{6}h^3 - \frac{1}{120}h^5$$

$$\begin{cases} f(x) = \sin x \xrightarrow{x_0=0} 0 \\ f'(x) = \cos x \xrightarrow{x_0=0} 1 \\ f''(x) = -\sin x \xrightarrow{x_0=0} 0 \\ f'''(x) = -\cos x \xrightarrow{x_0=0} -1 \\ f^{IV}(x) = \sin x \xrightarrow{x_0=0} 0 \\ f^{V}(x) = -\cos x \xrightarrow{x_0=0} -1 \end{cases}$$

$$f^{VI}(x) = \sin x \xrightarrow{x_0=0} 0$$

$$\sin x \approx x - \frac{x^3}{3!} - \frac{x^5}{5!}$$

$$R_5 = \frac{-\sin(\theta h)}{6!}h^6 \quad \theta \in [0,1]$$

$$R_5 = \frac{-\sin(\theta x)}{6!}x^6 \quad \theta \in [0,1]$$

* napako ocenimo za $x \in [0, \frac{\pi}{2}]$

≥ 0 za $x \in [0, \frac{\pi}{2}]$

$$|R_5| = \left|\frac{-\sin(\theta h)}{6!}x^6\right| = \frac{\sin(\theta x)}{6!}x^6 \leq \frac{1}{6!}x^6 \leq \frac{1}{6!}\left(\frac{\pi}{2}\right)^6 \approx 0{,}02$$

RISANJE FUNKCIJ

$$f(x) = \frac{x^3}{(x+1)^2}$$

① $D_f: \mathbb{R} - \{-1\}$

② Zv: $y=0$

③ A: $x^3 : (x^2+2x+1) = x-2 = y$

$$\begin{array}{l} x^3+2x^2+x \\ \hline -2x^2-x \\ -2x^2-4x-2 \\ \hline 3x+2 \rightarrow \text{OSTANEK} \end{array}$$

$3x+2=0 \rightarrow x=-\frac{2}{3}$

$y=-\frac{8}{3}$

$T(-\frac{2}{3}, -\frac{8}{3})$

KONKAVNOST

f konkavna

f' pada

$f'' < 0$

* $f''=0$ PREVOJ

KONVEKSNOST

f konveksna

f' narašča

$f'' > 0$

④ ODVOD

$$f'(x) = \frac{3x^2(x+1)^2 - x^3 \cdot 2(x+1) \cdot 1}{(x+1)^4}$$

$$f'(x) = \frac{(\cancel{x+1})(3x^2(x+1) - 2x^3)}{(x+1)^{\cancel{4}}}$$

$$f'(x) = \frac{3x^2(x+1) - 2x^3}{(x+1)^3}$$

STACIONARNE TOČKE $f'(x)=0$

$0 = 3x^2(x+1) - 2x^3 \qquad x \neq 1$

$0 = x^2(x+3)$

$x_{1,2}=0$

$f(0)=0$

$T_1(0,0) \leftarrow$ prevoj

$x_3=-3$

$f(-3) = -\frac{27}{4}$

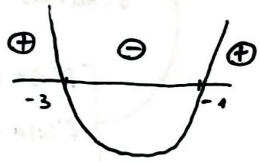
$T_2(-3, -\frac{27}{4}) \leftarrow$ lokalni MAX

⑤ PADANJE, NARAŠČANJE

$f'(x) > 0 \quad f$ narašča

$$\overset{\oplus}{\frac{x^2 (x+1)(x+3)}{(x+1)^3}}_{\oplus} > 0 \Leftrightarrow (x+1)(x+3) > 0 \Leftrightarrow x \in (-\infty, -3) \cup (-1, \infty)$$

$f'(x) < 0 \quad f$ pada $\Leftrightarrow x \in (-3, -1)$



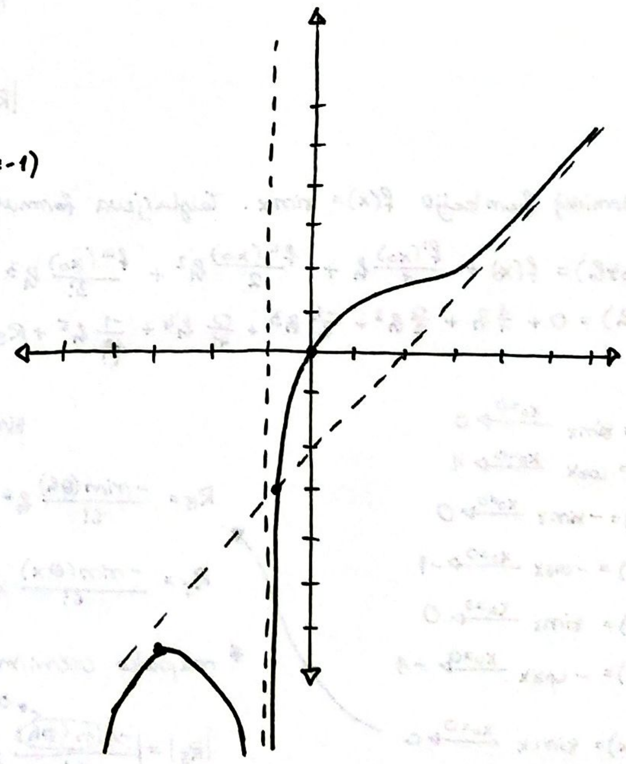
⑥ KONVEKSNOST, KONKAVNOST, PREVOJ

$$f''(x) = \frac{6x}{(x+1)^4}$$

↳ $f''(x)=0 \Leftrightarrow 6x=0 \Leftrightarrow x=0 \leftarrow$ PREVOJ

↳ $f''(x) > 0$ (konveksna) $\Leftrightarrow \frac{6x}{(x+1)^4} > 0 \Rightarrow x > 0$

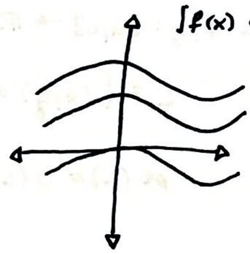
↳ $f''(x) < 0$ (konkavna) $\Leftrightarrow \frac{6x}{(x+1)^4} < 0 \Rightarrow x < 0 \quad (x \neq -1)$



(12) INTEGRALI

NEDOLOČENI INTEGRALI

- $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$

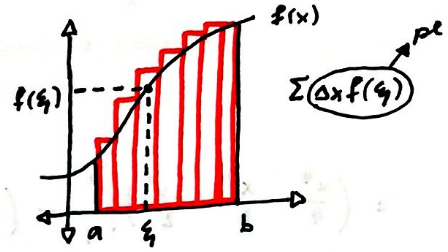


$\int f(x)\,dx = F(x) + c$
$(c \in \mathbb{R})$

OSNOVNI NEDOLOČNI INTEGRALI

- $\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad (\alpha \neq -1)$
- $\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \ln|x| + c \quad (x \neq 0)$
- $\int e^{x}\,dx = e^{x} + c$
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int a^{x}\,dx = \frac{a^{x}}{\ln a} + c$
- $\int \frac{1}{\sin^2 x}\,dx = -\cot x + c$
- $\int \frac{1}{\cos^2 x}\,dx = \mathrm{tg}\,x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c \quad (x \in (-1,1))$
- $\int \frac{1}{1+x^2}\,dx = \mathrm{arctg}\,x + c$

DOLOČENI INTEGRALI



1) če je funkcija f integrabilna na $[a,c]$ in $[c,b]$, potem je integrabilna tudi na $[c,b]$ – in obratno:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

2) če sta funkciji f in $g: [a,b] \to \mathbb{R}$ integrabilni in $\lambda \in \mathbb{R}$, sta integrabilni tudi funkciji $f+g$ in λf

- $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$
- če je $f(x) \leq g(x)$ za vse $x \in [a,b]$ potem: $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

3) za $a < b$ definiramo $\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$ in $\int_a^a f(x)\,dx := 0$

4) če je funkcija $f: [a,b] \to \mathbb{R}$ integrabilna, je integrabilna tudi $|f|$ in velja ocena:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

5) NEWTON – LEIBNIZOV ZAKON

naj bo ~~f(x)~~ $f: [a,b] \to \mathbb{R}$ zvezna funkcija in definirajmo

$$F(x) := \int_a^x f(t)\,dt \quad \text{za } x \in [a,b]$$

* POSLEDICA: naj bo zvezna na $[a,b]$, Φ odvedljiva na $[a,b]$ in $\Phi'(x) = f(x)$, za vsak $x \in [a,b]$, potem:

$$\int_a^b f(x)\,dx = \Phi(b) - \Phi(a) \equiv \left[\Phi(x)\right]_a^b \equiv \Phi(x)\Big|_a^b$$

funkcijo Φ imenujemo PRIMITIVNA FUNKCIJA funkcije f na intervalu $[a,b]$

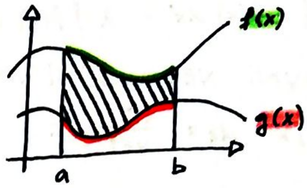
• $\int_1^2 (x^2+x^{-4})\,dx = \left(\frac{x^3}{3}+\frac{x^{-3}}{-3}\right)\Big|_1^2 = \left(\frac{x^3}{3}-\frac{x^{-3}}{3}\right)\Big|_1^2 = \left(\frac{2^3}{3}-\frac{2^{-3}}{3}\right)-\left(\frac{1^3}{3}-\frac{1^{-3}}{3}\right) = \frac{21}{8}$

PLOŠČINA LIKA

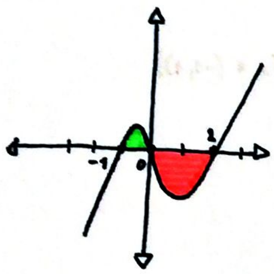
naj bosta $f, g : [a,b] \to \mathbb{R}$ zvezni in naj velja $f(x) \geq g(x)$ za vsak $x \in [a,b]$. Potem je ploščina lika

$$L = \{(x,y) \in \mathbb{R} \mid a \leq x \leq b,\ g(x) \leq y \leq f(x)\}$$ enaka

$$pl(L) = S(L) = \int_a^b (f(x) - g(x))\,dx$$



• med krivuljo $y = x^3 - x^2 - 2x$ in x-osjo



① PRESEČIŠČE med krivuljama

$f(x) = y = x^3 - x^2 - 2x$
$g(x) = y = 0$
$\rightarrow 0 = x(x-2)(x+1)$

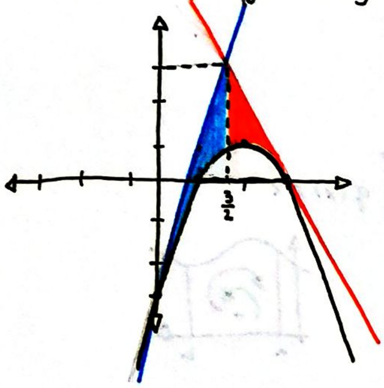
$x_1 = 0$
$x_2 = 2$
$x_3 = -1$

② LOČI PLOŠČINI ⊕ IN ⊖ DELA

$$pl = \int_{-1}^{0} (\underbrace{(x^3-x^2-2x)}_{f(x)} - \underbrace{0}_{g(x)})\,dx + \int_0^2 (\underbrace{0}_{g(x)} - \underbrace{(x^3-x^2-2x)}_{f(x)})\,dx =$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right)\Big|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right)\Big|_0^2 = -\left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right) - \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) = \frac{37}{12}$$

• ploščina lika, ki ga tvorijo parabola $y = -x^2 + 4x - 3$ ter njeni tangenti v točkah $(0, y_1)$ in $(3, y_2)$



① TOČKI

$y(0) = -3 \rightarrow P_1(0,-3)$
$y(3) = 0 \rightarrow P_2(3,0)$

② TANGENTI

1) v $(0, y_1) = (0, -3)$

$y = k_1 x + m_1$
$f'(0) = k_1 = -2\cdot 0 + 4 = 4$
$-3 = 4\cdot 0 + m_1 \rightarrow m_1 = -3$

$y_1 = 4x - 3$

2) v $(3, y_2) = (3, 0)$

$y = k_2 x + m_2$
$f'(3) = k_2 = -2\cdot 3 + 4 = -2$
$0 = -2\cdot 3 + m_2 \rightarrow m_2 = 6$

$y_2 = -2x + 6$

③ PRESEČIŠČE TANGENT

$4x - 3 = 2x + 6$
$x = \frac{3}{2}$

④ PLOŠČINA

$pl = pl_1 + pl_2 =$

$= \int_0^{\frac{3}{2}} ((4x-3) - (-x^2+4x-3))\,dx + \int_{\frac{3}{2}}^{3} ((-2x+6) - (-x^2+4x-3))\,dx$

$= \frac{9}{4}$

VPELJAVA NOVE SPREMENLJIVKE

- NEDOLOČNI INTEGRAL

Če je $F(x) = \int f(x)\,dx$ in je $x = x(t)$ odvedljiva funkcija, potem velja

$$\boxed{F(x(t)) = \int f(x(t)) \cdot x'(t)\,dt}$$

- DOLOČENI INTEGRALI

Če je $f(x)$ zvezna na intervalu $[a,b]$, funkcija $x(t)$ zvezno odvedljiva na $[a,b]$ in $x \in ([\alpha,\beta]) \subset [a,b]$, potem velja

$$\boxed{\int_{x(\alpha)}^{x(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(x(t)) \cdot x'(t)\,dt}$$

- $\int (4x-3)^5\,dx = \int t^5\,\frac{1}{4}dt = \frac{1}{4}\int t^5\,dt = \frac{1}{4}\,\frac{t^6}{6} = \frac{1}{4}\,\frac{(4x-3)^6}{6} + C$

$t = 4x-3$

ODVOD PO t — ODVOD PO X

$dt = 4\,dx$

$\frac{1}{4}dt = dx$

- $\int \frac{1}{(9x^2+4)}\,dx = \int \frac{1}{4(\frac{9x^2}{4}+1)}\,dx = \int \frac{1}{4((\frac{3}{2}x)^2+1)}\,dx = \frac{1}{4}\int \frac{1}{t^2+1}\,\frac{2}{3}dt = \frac{1}{4}\cdot\frac{2}{3}\int \frac{1}{t^2+1}\,dt = \frac{1}{6}\,\mathrm{arctg}(\frac{3}{2}x) + C$

ŽELIMO SE ZNEBITI 4

$t = \frac{3}{2}x$

$dt = \frac{3}{2}dx$

$\frac{2}{3}dt = dx$

* Če je ulomek pomisli na $\mathrm{arctg}\,x$

- $\int \frac{2x}{\sqrt{x^2+3}}\,dx = \int \frac{1}{\sqrt{t}}\,dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x^2+3} + C$

$t = x^2+3$

$dt = 2x\,dx$

- $\int_1^4 \frac{\sqrt{x}+\ln x}{x}\,dx = \int_1^4 \left(\frac{\sqrt{x}}{x} + \frac{\ln x}{x}\right)dx = \int_1^4 x^{-\frac{1}{2}}dx + \int_1^4 \frac{\ln x}{x}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\Big|_1^4 + \int_1^4 t\,dt = \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{\ln^2 x}{2}\right)\Big|_1^4 =$

$t = \ln x$

$dt = \frac{1}{x}dx$

$= \left(2\sqrt{x} + \frac{\ln^2 x}{2}\right)\Big|_1^4$

$= \left(2\sqrt{4} + \frac{\ln^2 4}{2}\right) - \left(2\sqrt{1} + \frac{\ln^2 1}{2}\right)$

$= \left(2\sqrt{4} + \frac{\ln^2 2^2}{2}\right) - \left(2\sqrt{1} + \frac{\ln^2 1}{2}\right) = 4 + \frac{2\cdot\ln^2 2}{2} - 2 - \cancel{\frac{\ln^2 1}{2}} = 4 + \ln^2 2 - 2 = 2 + \ln^2 2$

INTEGRIRANJE PO DELIH (PER PARTES)

- NEDOLOČNI INTEGRALI

  Če sta u in v odvedljivi funkciji, velja:

  $$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx \quad \text{oz. krajše} \quad \boxed{\int u\,dv = uv - \int v\,du}$$

- DOLOČENI INTEGRALI

  Če sta $u, v : [a,b] \rightarrow \mathbb{R}$ zvezno odvedljivi, potem velja

  $$\int_a^b u(x)v'(x)\,dx = \left(u(x)v(x)\right)\Big|_a^b - \int_a^b u'(x)v(x)\,dx \quad \text{oz. krajše} \quad \boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

- $$\int \underbrace{(x+1)}_{u}\underbrace{e^x\,dx}_{dv} = \overset{u}{(x+1)}\cdot \overset{v}{e^x} - \int \overset{v}{e^x}\,\overset{du}{dx} = xe^x + e^x - e^x + C = xe^x + C$$

  $u = x+1$, $du = dx$ ODVEDEŠ

  $dv = e^x dx$, $v = \int e^x dx = e^x$ INTEGRIRAŠ

- $$\int \underbrace{\operatorname{arctg} x}_{u}\,\underbrace{dx}_{dv} = x\cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2}\,dx = x\operatorname{arctg} x - \frac{1}{2}\int \frac{1}{t}\,dt = x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C$$

  $u = \operatorname{arctg} x$, $du = \frac{1}{1+x^2}dx$

  $dv = dx$, $v = \int dx = x$

  NOVA SPREMENLJIVKA

  $t = 1+x^2$

  $dt = 2x\,dx$

- $$\int_0^{\frac{1}{2}} \underbrace{\arcsin x}_{u}\,\underbrace{dx}_{dv} = (\arcsin x \cdot x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x\,\frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin\frac{1}{2} - \int_0^{\frac{1}{2}} -\frac{1}{t^{\frac{1}{2}}}\,\frac{dt}{2} = \frac{1}{2}\arcsin\frac{1}{2} - \sqrt{1-x^2}\Big|_0^{\frac{1}{2}}$$

  $$= \frac{1}{2}\arcsin\frac{1}{2} + \sqrt{1-\frac{1}{4}} - \sqrt{1}$$

  $$= \frac{\pi + 3\sqrt{3} - 12}{12}$$

  $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}dx$

  $dv = dx$, $v = \int dx = x$

  $t = 1-x^2$

  $dt = -2x\,dx$

- $$\int \frac{1}{x^2-2x+3}\,dx = \int \frac{1}{(x-1)^2-1+3}\,dx = \int \frac{1}{(x-1)^2+2}\,dx = \frac{1}{2}\int \frac{1}{\left(\frac{x-1}{\sqrt{2}}\right)^2+1}\,dx = \frac{1}{2}\int \frac{1}{t^2+1}\,dt\sqrt{2} =$$

  $$= \frac{\sqrt{2}}{2}\operatorname{arctg} t + C$$

  $$= \frac{\sqrt{2}}{2}\operatorname{arctg}\left(\frac{x-1}{\sqrt{2}}\right) + C$$

  * dopolnjevanje do popolnega kvadrata

  Želimo se znebiti 2

  $t = \frac{x-1}{\sqrt{2}} = \frac{x}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

  $dt = \frac{dx}{\sqrt{2}}$

  $\sqrt{2}\,dt = dx$

- INTEGRIRANJE RACIONALNIH FUNKCIJ

  1) OKRAJŠAJ

  2) Če je v ulomku $\int \frac{p(x)}{a(x)}\,dx$ stopnja $p(x)$ večja ali enaka stopnji $a(x)$ $\rightarrow$ ULOMEK DELIMO

  3) RAZSTAVIMO IMENOVALEC / ULOMEK

- $$\int \frac{1}{x+x^2}\,dx = \int \frac{1}{x(x+1)}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C = \ln\frac{|x|}{|x+1|} + C$$

  $$* \int \frac{dx}{x+a} = \ln|x+a| + C$$

  3) RAZBIJANJE ULOMKA

  $$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{Ax+A+Bx}{x(x+1)} = \frac{(A+B)x + A}{x(x+1)}$$

  $A = 1 \rightarrow \frac{1}{x}$

  $A+B = 0 \rightarrow \frac{-1}{x+1}$

  $B = -1$

$$* \quad \frac{1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

• $\int_2^3 \frac{x^2+1}{x-1}\,dx = \int_2^3 \left(x+1+\frac{2}{x-1}\right)dx = \left(\frac{x^2}{2}+x+2\ln|x-1|\right)\Big|_2^3 = \frac{3^2}{2}+3+2\ln|3-1|-\left(\frac{2^2}{2}+2+2\ln|2-1|\right) =$

$= \frac{7}{2}+2\ln 2$

2) v imenovalcu manjša potenca kot v števcu → DELIMO

$$
\begin{array}{l}
(x^2+1):(x-1) = x+1 \\
\underline{x^2-x} \\
x+1 \\
\underline{x-1} \\
2
\end{array}
$$

$$\boxed{\begin{array}{l} \cos^2 x = \frac{1}{2}(1+\cos(2x)) \\ \sin^2 x = \frac{1}{2}(1-\cos(2x)) \end{array}}$$

• $\int \sin^2 x \cdot \cos^2 x\,dx = \int \frac{1}{2}(1-\cos(2x))\cdot\frac{1}{2}(1+\cos(2x))\,dx = \frac{1}{4}\int(1-\cos(2x))(1+\cos(2x))\,dx =$

$= \frac{1}{4}\int(1-\cos^2(2x))\,dx = \frac{1}{4}\int\left(1-\frac{1}{2}(1+\cos(4x))\right)dx = \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{2}\int \cos t\ \frac{1}{4}dt\right) = \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{8}\int \cos t\,dt\right) =$

$= \frac{1}{4}\cdot\frac{1}{2}x-\frac{1}{4}\frac{1}{8}\sin(4x)+c$

$= \frac{1}{8}x-\frac{1}{32}\sin(4x)$

$t = 4x$

$dt = 4dx$

$\frac{1}{4}dt = dx$