BOŠTJAN BRANK 408
bbrank@fgg.uni-lj.si

NUMERIČNE METODE

TO 10^{15} - 11^{45}, PREDAVALNICA

→ PRIJAVA NA SPLETNO UČILNICO

PO (VAJE) 12^{00} - 14^{00} (12^{00} - 13^{30})

→ IZPIT (KOT PRI PK) → 1. IN 2. ROK
L SEMINAR (SE ODDAJA NA SPLETNI UČILNICI)
L RAČUNSKI DEL (RK)
L TEORIJA (PISNO)

*) KE ZA PLOŠČE (PROSOJNICE NA SPLETNI UČILNICI)
L PREDAVANJE 7.1.2014

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NUMERIČNE METODE

NUMERIČNE METODE SO PONAVADI MIŠLJENE KOT DEL MATEMATIKE (= NUMERIČNA MATEMATIKA)

↳ REŠEVANJE SISTEMA LINEARNIH ENAČB

↳ NUMERIČNA INTEGRACIJA

↳ NUMERIČNA INTEGRACIJA DIFERENCIALNIH ENAČB

↳ PROBLEM LASTNIH VREDNOSTI

↳ INTERPOLACIJA IN EKSTRAPOLACIJA

↳ REŠEVANJE SISTEMA NELINEARNIH ENAČB

→ NUM. MAT. NAREDI OCENO NAPAKE.

NUMERIČNE METODE V TEHNIKI (GRADBENIŠTVO)

VZAMEMO METODE NUMERIČNE MATEMATIKE IN JIH PRILAGODIMO NAŠIM PROBLEMOM

↳ $\boxed{\text{MKE}}$ (METODA KONČNIH ELEMENTOV) ↳ $\boxed{\text{LINEARNA MKE}}$

- ↳ INTERPOLACIJA
- ↳ NUM. INTEGRACIJA
- ↳ REŠEVANJE SISTEMA LIN. ENAČB

↳ METODA KONČNIH DIFERENC

↳ METODA ROBNIH ELEMENTOV

↳ METODA KONČNIH VOLUMNOV

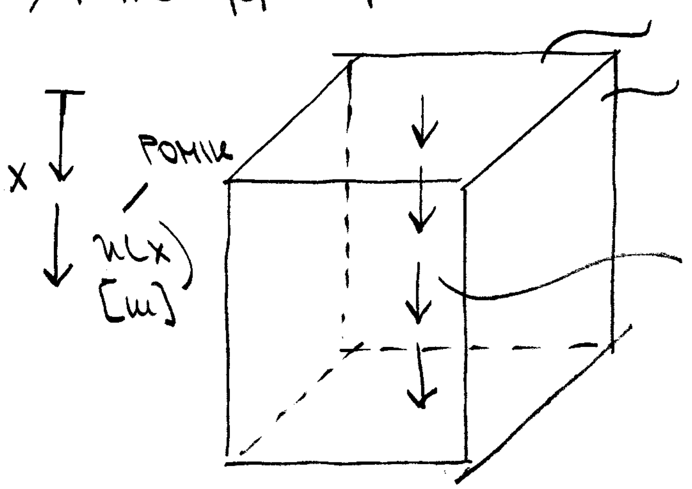
↳ BREZMREŽNE METODE

A: NEKATERI ENODIMENZIONALNI PROBLEMI V GRADBENIŠTVU (1D)

NJIHOV MATEMATIČNI OPIS V OBLIKI DIFERENCIALNIH ENAČB (DE)

A.1. 1D ELASTIČNOST (ELASTIČNA PALICA)

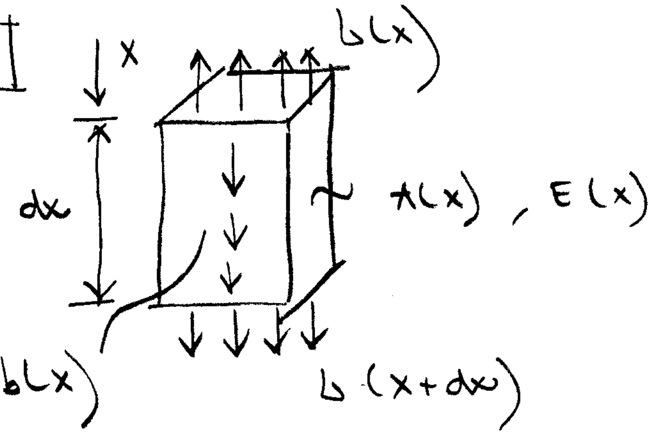
→ POTREBUJEMO MODEL



MODEL (MATEMATIČNE ENAČBE) DOBIMO PRI TEORIJI ELASTIČNOSTI S <u>3 TIPI ENAČB</u>:

1. RAVNOTEŽNE ENAČBE (RE)
2. KINEMATIČNE ENAČBE (KE)
3. KONSTITUTIVNE ENAČBE (KOE)

$\boxed{\text{RE}}$



1D PROBLEM:

GIBANJE SAMO V 1 SMERI (x)

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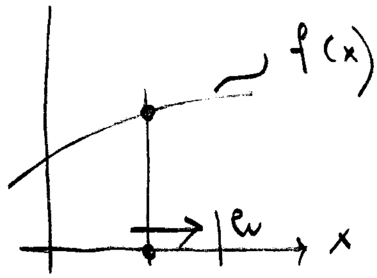
TAYLOR. VRSTA: $\sigma(x)+\frac{d\sigma(x)}{dx}dx+\underbrace{O(dx^2)}_{\text{OSTANEK}}$

$$-\sigma(x)A(x)+\sigma(x+dx)\,\underbrace{A(x+dx)}_{A(x)+\frac{dA(x)}{dx}dx+O(dx^2)}+\underbrace{\int_x^{x+dx} b(\bar{x})A(\bar{x})\,d\bar{x}}_{\approx\, b(x+\frac{dx}{2})A(x+\frac{dx}{2})}=0$$

TAYLORJEVA VRSTA:

$$f(x+h)=f(x)+\frac{df(x)}{dx}h+\frac{1}{2}\frac{d^2f(x)}{dx^2}h^2+\dots$$



$$b\left(x+\frac{dx}{2}\right)=b(x)+\frac{db(x)}{dx}dx+O(dx^2)$$

$$A\left(x+\frac{dx}{2}\right)=A(x)+\frac{dA(x)}{dx}dx+O(dx^2)$$

$O(dx^2)$ – OSTANEK PRI INFINITEZIMALNO MAJHNEM DELCU LAHKO ZANEMARIMO, DELIMO Z dx IN DOBIMO:

$$\boxed{\frac{d}{dx}[\underbrace{\sigma(x)A(x)}_{\text{OSNA SILA}}]+b(x)A(x)=0}$$ RE

VE: $$\boxed{\varepsilon(x)=\frac{du(x)}{dx}}$$

KoE: $$\boxed{\sigma(x)=E(x)\cdot\varepsilon(x)}$$

ČE VSE 3 ENAČBE SESTAVIMO SKUPAJ...

$$\boxed{\frac{d}{dx}\left[E(x)\,\frac{du(x)}{dx}\,A(x)\right] + b(x)\,A(x) = 0}$$

└ DE RAZTEGOVANJA / KRČENJA PALICE

REZULTAT: POMIKI, NAP., DEF.

ZA $E(x) = E$, $A(x) = A$, $b(x) = b$, DOBIMO:

$$\boxed{EA\,\frac{d^2 u(x)}{dx^2} + bA = 0}$$

DE 2. REDA

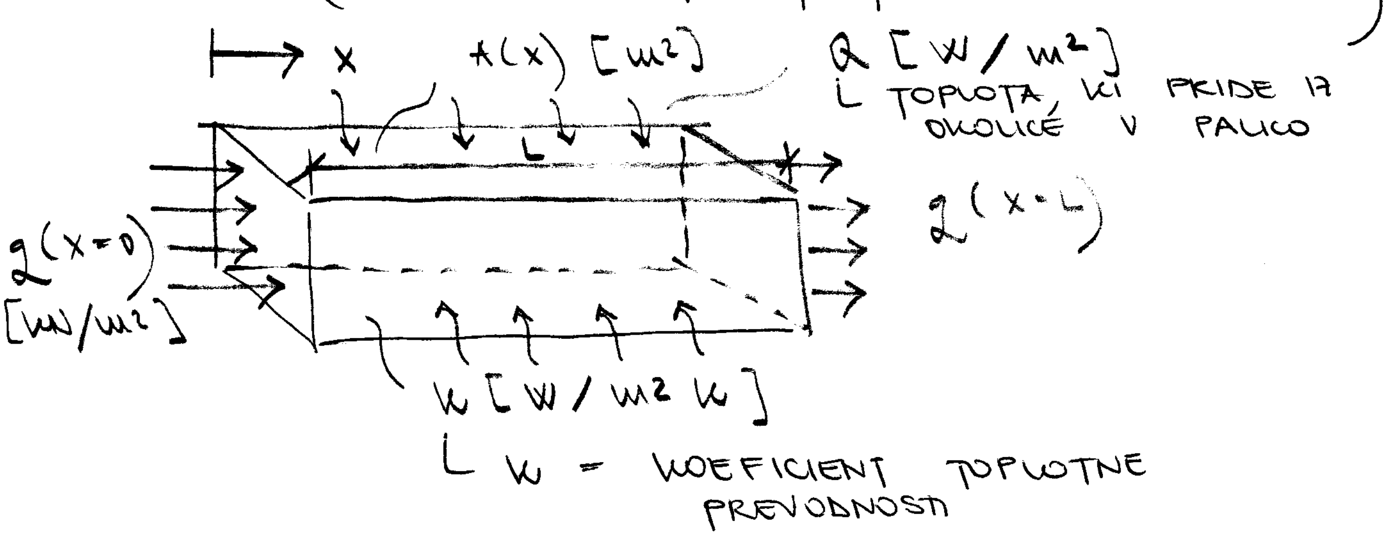
└ DE RAZTEGOVANJA / KRČENJA PALICE S KONSTANTNIMI A, E, b.

→ PROBLEME OPISUJEJO DE

↳ PALICA DE 2. REDA

↳ NOSILEC, PLOŠČA DE 4. REDA

A.2. PREHAJANJE TOPLOTE V 1D
(STACIONARNO PREVAJANJE - NEODVISNO OD ČASA)



IŠČEMO MODEL (ENAČBE), KI OPISUJEJO TAKŠNO PREHAJANJE TOPLOTE.

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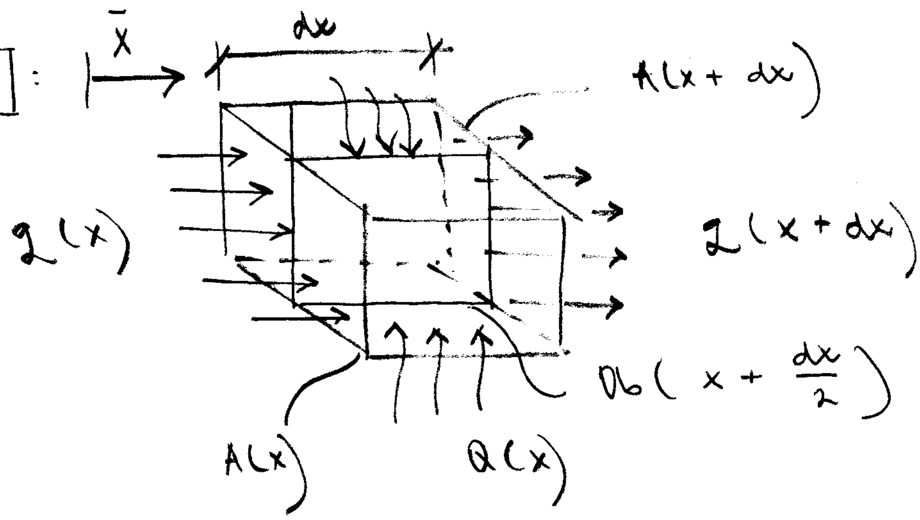
$\boxed{RE}$:



$$q(x)\,A(x) + \int_{S} Q(\bar{x})\,dS - \underbrace{q(x+dx)}\,\underbrace{A(x+dx)} = 0$$

S = POVRŠINA

$$q(x+dx) = q(x) + \frac{dq(x)}{dx}dx + O(dx^2)$$
$$A(x+dx) = A(x) + \frac{dA(x)}{dx}dx + O(dx^2)$$

POVRŠINA $S \approx \underbrace{Q\left(x+\frac{dx}{2}\right)}_{Q(x)+\frac{dQ(x)}{dx}dx+O(dx^2)}\ \underbrace{Ob\left(x+\frac{dx}{2}\right)}_{Ob(x)+\frac{dOb(x)}{dx}dx+O(dx^2)} dx$

Ob: OBSEG

ZANEMARIM OSTANEK $O(dx^2)$, DELIM Z dx, DOBIM:

$$\boxed{-\frac{d}{dx}\left[\,q(x)\,A(x)\,\right] + \bar{Q}(x) = 0}$$

L (RE) PREHAJANJA TOPLOTE V 1D

$$\bar{Q}(x) = Q(x)\,Ob(x)$$

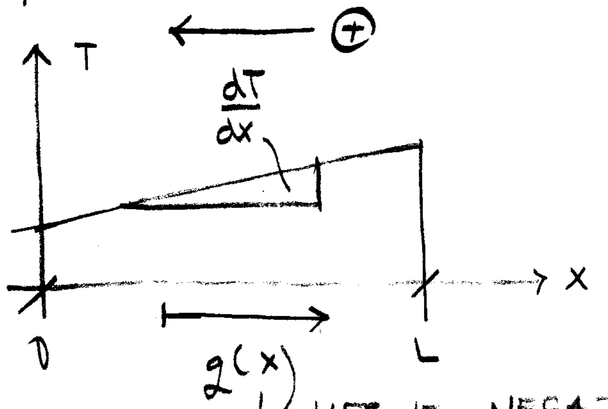
KE NI, IMAMO PA KOE:

TEMPERATURA [K]

$\boxed{KOE}$:
$$\boxed{q(x) = -\,k(x)\,\frac{dT(x)}{dx}}$$

$\left[\frac{W}{m^2 K}\right]$

k !



TOPLOTNI TOK

KER JE NEGATIVEN

→ ČE OBE ENAČBI ZDRUŽIMO, DOBIMO DE PREHAJANJA TOPLOTE V 1D

$$\boxed{\frac{d}{dx}\left[k(x)\,\frac{d\,T(x)}{dx}\,A(x)\right]+\bar{Q}(x)=0}$$

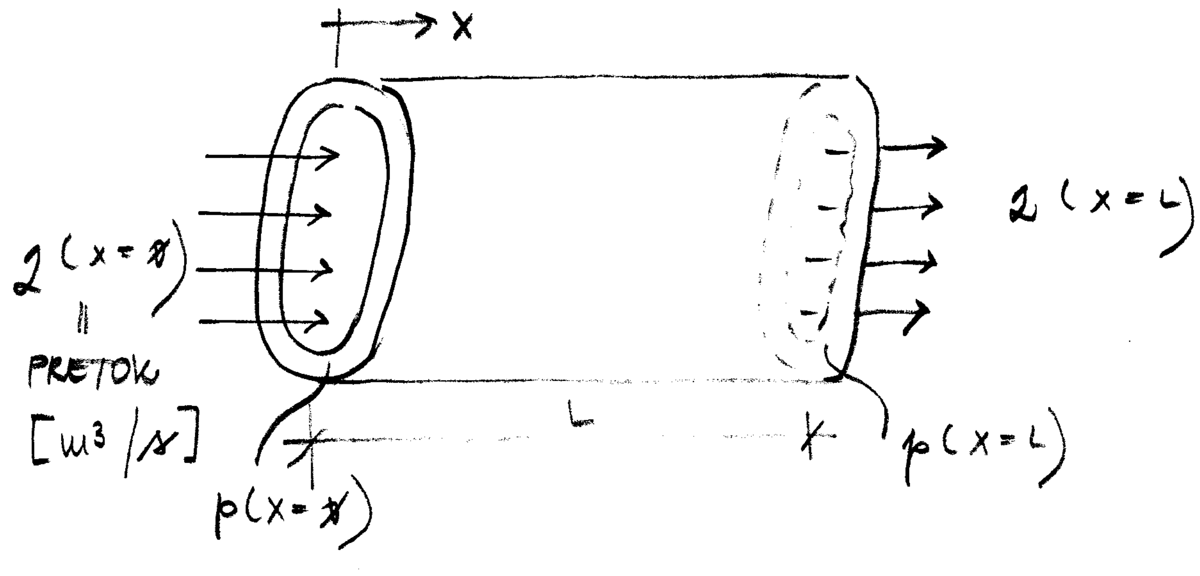
ČE $k(x)=k$, $A(x)=A$, $\bar{Q}(x)=Q$, IMAMO:

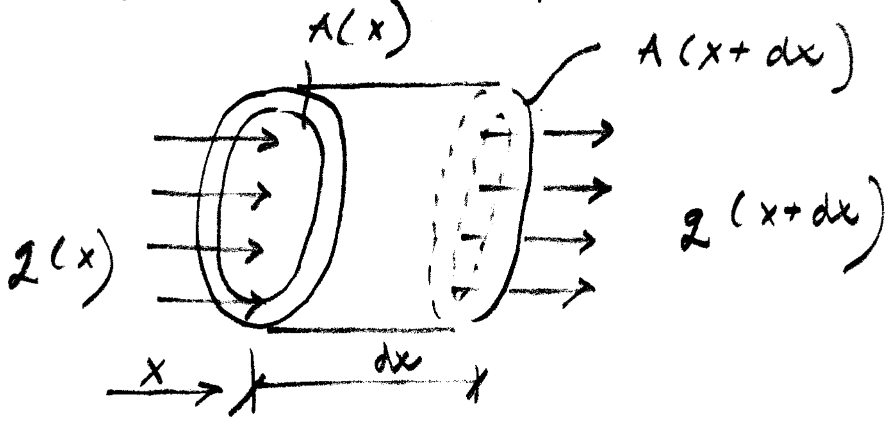
$$\boxed{kA\,\frac{d^2T(x)}{dx^2}+\bar{Q}=0}$$

└ DE PREHAJANJA TEMPERATURE PO 1D

PODOBNA DE KRČENJA / RAZTEZANJA PALICE, NAMESTO POMIKA → TEMPERATURA

A.3. TOK PO CEVI POD PRITISKOM (STACIONAREN TOK)

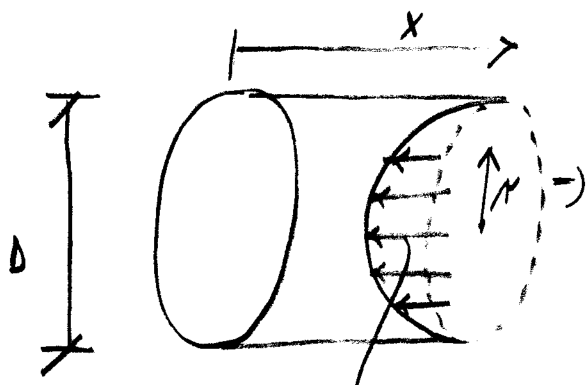




$$q(x)\,A(x) - \underbrace{q(x+dx)}_{q(x)+\frac{dq(x)}{dx}dx+O(dx^2)}\ \underbrace{A(x+dx)}_{A(x)+\frac{dA(x)}{dx}dx+O(dx^2)} = 0$$

DOBIMO: $\boxed{\frac{d}{dx}\left[\,q(x)\,A(x)\,\right] = 0}$ RE

KOE IZPELJEMO NA NASLEDNJI NAČIN:



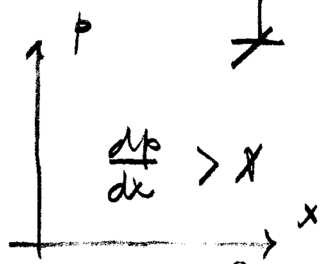
$\Rightarrow r \in [0, \frac{D}{2}]$

PRITISK

$$v(r) \Rightarrow v(r) = \ominus \frac{1}{4\mu}\left(1-\left(\frac{2r}{D}\right)^2\right)\frac{dp}{dx}$$

VISKOZNOSTNI KOEFICIENT

VODA TEČE OD VEČJEGA PRITISKA PROTI MANJŠEMU



$$q(x) = \underbrace{\int_{A(x)} v(x)\,dA}_{-\frac{\pi D(x)}{128\mu}\frac{dp}{dx}}$$

$$\frac{\pi D(x)}{128\mu} = k(x) \rightarrow \text{"UPORNOST"}$$

$$q(x) = -k(x)\,\frac{dp(x)}{dx}$$

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ČE ENAČBI ZDRUŽIM:

$$\boxed{\frac{d}{dx}\left[-k(x)\,\frac{dp(x)}{dx}\,A(x)\right]=0}$$

↳ DE TOKA V CEVI

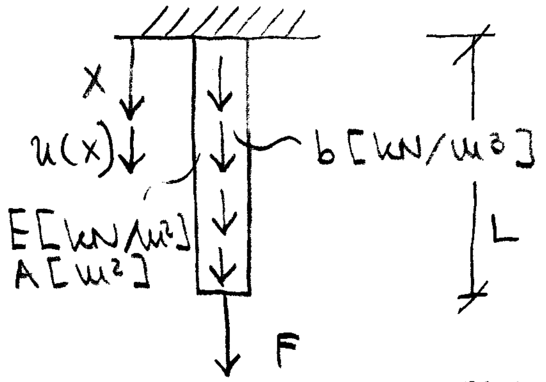
B. 1D ROBNI PROBLEMI

(ZA PRIMERE POD A)

→ KAJ JE ROBNI PROBLEM?

- DIF. ENAČBA, KI OPISUJE NEK POJAV + ROBNI POGOJI

B.1. 1D ELASTIČNOST



ROBNI PROBLEM:

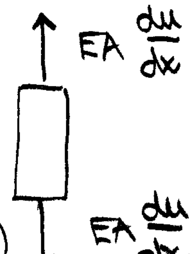
→ ČE POZNAŠ E, A, b, L, F, POIŠČI TAKŠEN u(x), DA BO VELJALA DIF. ENAČBA

$$\boxed{EA\,\frac{d^2u(x)}{dx^2}+bA=0} \text{ ZA } \forall x\in(0,L)$$

↱ ROBOVI IZLOČENI

TER DA BOSTA IZPOLNJENA ROBNA POGOJA:

$$\boxed{u(0)=0}\ ① \quad \text{IN} \quad \boxed{\underbrace{EA\,\frac{du(x)}{dx}}_{\text{OSNA SILA}}\Bigg|_{x=L}=F.}\ ②$$



↳① ČE PREDPIŠEMO VREDNOST FUNKCIJE KI JO IŠČEMO, JE TO BISTVENI (DIRICHLETOV) POGOJ.

↳② ČE PREDPIŠEMO VREDNOST ODVODA FUNKCIJE, KI JO IŠČEMO, JE TO NARAVNI (NEWMANNOV) POGOJ.

→ ALI LAHKO TA ROBNI PROBLEM REŠIMO? DA. KER JE PROBLEM ENOSTAVEN.

↳ REŠITEV = ANALITIČNA

ZA KAKŠEN BOLJ KOMPLICIRAN ROBNI PROBLEM, NE MOREMO DOBITI REŠITVE.

→ ANALITIČNA REŠITEV:

$$EA \frac{du(x)}{dx} = -bAx + C_1$$

$$EA\, u(x) = -bA\frac{x^2}{2} + C_1 x + C_2$$

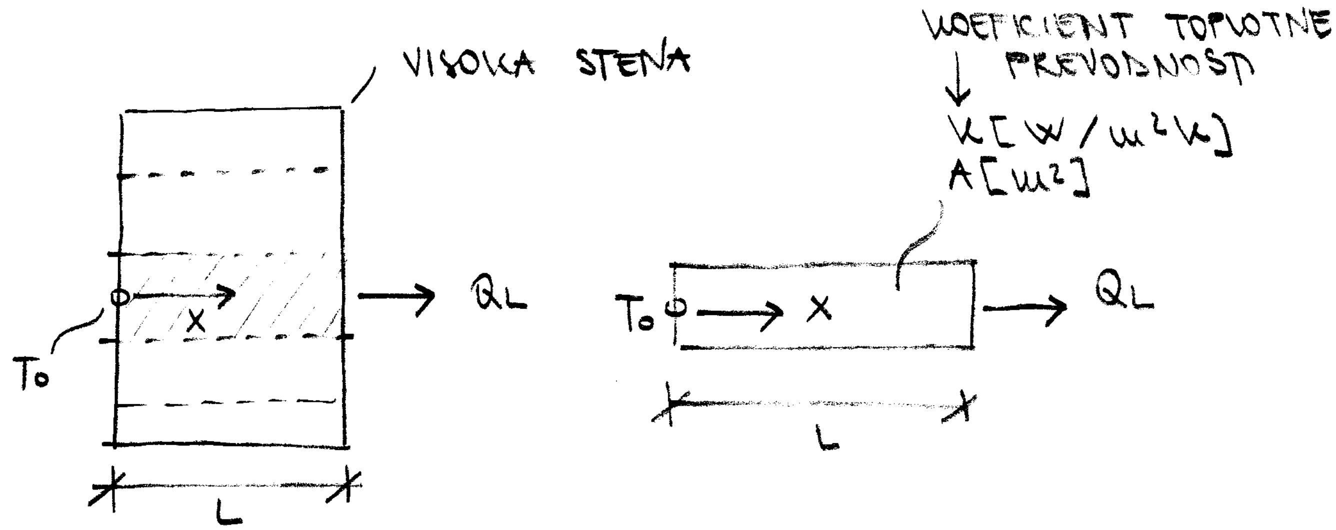
$$\begin{cases} x = 0 : & u(x) = 0 \Rightarrow C_2 = 0 \\ x = L : & EA \left.\frac{du(x)}{dx}\right|_{x=L} = F \Rightarrow -bA \cdot L + C_1 L - F \end{cases}$$

$$\Rightarrow C_1 = F + bA \cdot L$$

REŠITEV:

$$\boxed{EA\, u(x) = -bA\frac{x^2}{2} + (F + bAL)\, x}$$

B.2. 1D PREVAJANJE TOPLOTE



ROBNI PROBLEM:

OB ZNANIH k, A, L, T_0 IN Q_L DOLOČI TAKŠEN $T(x)$, DA BO ZADOŠČENA DIF. ENAČBA:

$$kA \frac{d^2T(x)}{dx^2} = 0 \quad \text{ZA } \forall x \in [0, L],$$

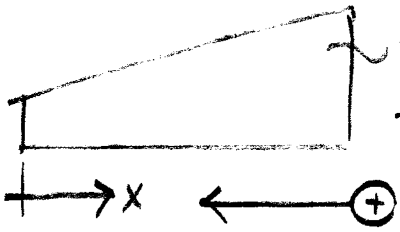
TER DA

↳ ROBOVI IZLOČENI!

BOSTA ZADOŠČENA ROBNA POGOJA: ① $T(0) = T_0$ IN ② $-kA \left.\frac{dT(x)}{dx}\right|_{x=L} = Q_L$.

① BISTVENI ROBNI POGOJ (FUNKCIJA JE PREDPISANA)

② NARAVNI ROBNI POGOJ (ODVOD F JE PREDPISAN)



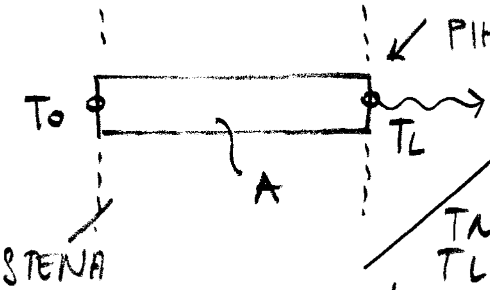
$T(x)$; $\frac{dT(x)}{dx} > 0$

TEMPERATURA TEČE OD VIŠJE T K NIŽJI (POZITIVNA SMER)

$$Q(x) = -kA\,\frac{dT(x)}{dx}$$

(*) TA ROBNI PROBLEM IMA TUDI MEŠANI ROBNI POGOJ, KO TOPLOTNI TOK POVEŽEMO S TEMPERATURO.

NPR.:



PIHA VETER MALO HLADI

$$\boxed{Q = \beta\,(T_L - T_a)\,A}$$

T_A = TEMP. ZRAKA (AMBIENTA)

T_L = TEMP. NA KONCU „PALICE"

NEWTONOV POGOJ

$\beta\,[W/m^2K]$

↳ KOEFICIENT KONVENCIJSKEGA PRESTOPA

$\rightarrow T_L = ?$

PRI PALICI VZMET

→ POIŠČIMO ANALITIČNO REŠITEV ZA PRVI ROBNI PROBLEM:

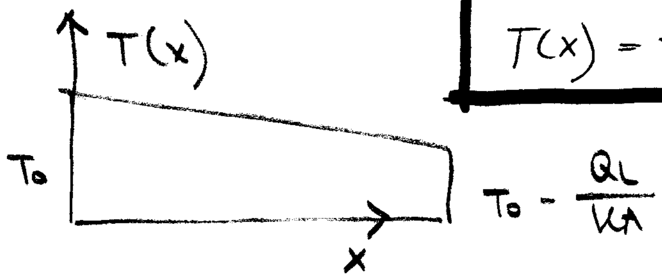
$$kA\,\frac{dT(x)}{dx} = C_1$$

$$kA\,T(x) = C_1 x + C_2$$

PRI: $\left[\begin{array}{ll} x = 0: & kA\,T_0 = C_2 \\ x = L: & kA\,\frac{dT(x)}{dx}\Big|_{x=L} = -Q_L = C_1 \quad \Rightarrow C_1 = -Q_L \end{array}\right.$

→ DOBIM REŠITEV: $T(x) = \frac{1}{kA}\left(-QL + kAT_0\right)$

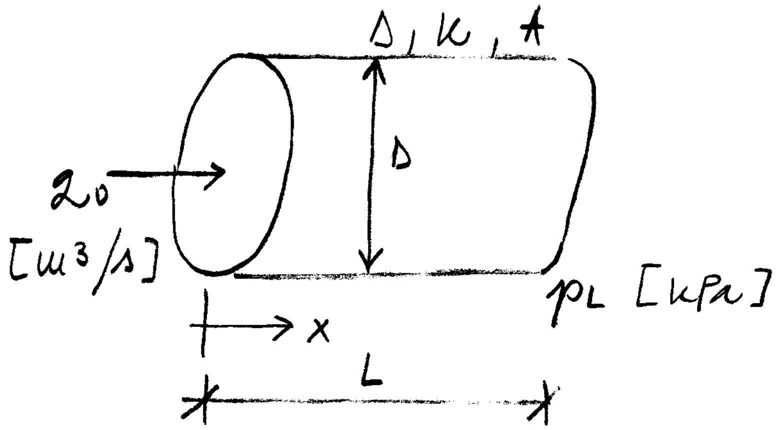
$$\boxed{T(x) = -\frac{QL}{kA}\cdot x + T_0}$$



B.3. TOK V CEVI

$k = \frac{\pi D^4}{128\,\mu}$ → VISKOZNOST TEKOČINE

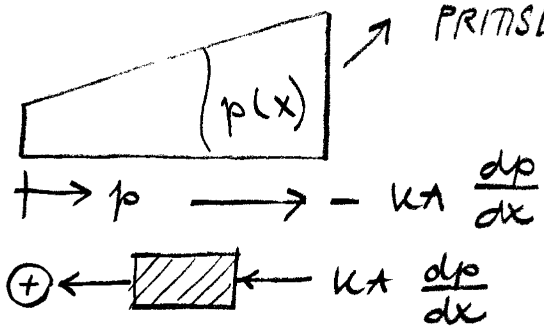
↳ "UPORNOST"



ROBNI PROBLEM PRI ZNANEM D, k, q_0 IN p_L, DOLOČI TAKŠEN $p(x)$ ZA $\forall\, x \in [0, L]$, DA BO VELJALO:

$Ak \frac{d^2 p(x)}{dx^2} = 0$, DA BO ZADOŠČENO ROBNIMA POGOJEMA:

$p(L) = p_L$ IN $-kA \frac{dp}{dx}\Big|_{x=0} = q_0$.



PRITISK TEČE OD VEČJEGA K MANJŠEMU

→ ANALITIČNO REŠITEV LAHKO DOLOČIMO:

$$Ak \frac{dp(x)}{dx} = C_1$$

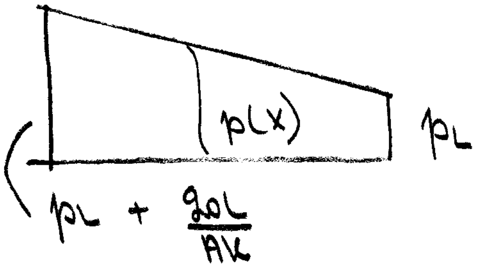
$$Ak\,p(x) = C_1 x + C_2$$

$\left[\, x = L: \; p(L) = p_L \Rightarrow Ak\,p_L = C_1 L + C_2 \right.$

$\left[\, x = 0: \; Ak \frac{dp(x)}{dx}\Big|_{x=0} = -q_0 = C_1 \right.$

→ $C_2 = AK p_L + g_0 L$

DOBIMO REŠITEV: $\boxed{p(x) = \frac{1}{AK}\left(-g_0 x + AK p_L + g_0 L\right)}$



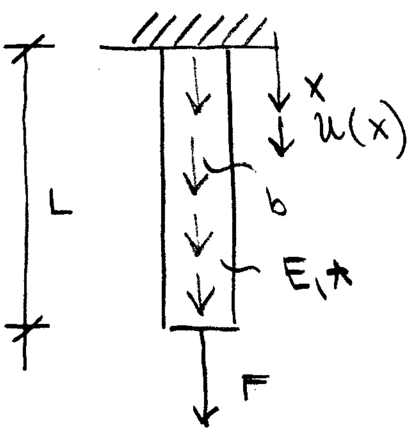
C. ŠIBKA OBLIKA ROBNEGA PROBLEMA

ČE ROBNEGA PROBLEMA NE MOREMO REŠITI ANALITIČNO, ENAČBE PREOBLIKUJEMO Z NAMENOM, DA JIH REŠIMO PRIBLIŽNO.

ENA MOŽNOST JE, DA NAPIŠEMO T.I. ŠIBKO OBLIKO ROBNEGA PROBLEMA.

TO POTEM REŠUJEMO PRIBLIŽNO.

C.1. ŠIBKA OBLIKA ZA 1D ELASTIČNOST



$$EA \frac{d^2 u(x)}{dx^2} + bA = 0 \qquad \text{DE}$$

$$u(0) = 0$$

$$EA \left.\frac{du(x)}{dx}\right|_{x=L} = F$$

IDEJA JE, DA DE NE ZADOSTI ZA $\forall x \in [0, L]$, AMPAK V NEKEM POVPREČENEM SMISLU NA CELEM OBMOČJU $(0, L)$.

→ NI TREBA, DA VSAKA TOČKA MIRUJE, POMEMBNO JE, DA SE VSE SKUPAJ NE PREMAKNEJO.

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$$\int_0^L \hat{w}\left(EA \frac{du^2(x)}{dx^2} + bA\right) dx = 0$$

TESTNA FUNKCIJA (= OBTEŽNA FUNKCIJA)

= POLJUBNA, EDINI POGOJ JE, DA JE 0 NA ROBU, NA KATEREM JE u PREDPISAN.

(V NAŠEM PRIMERU JE $u(x)$ PREDPISAN PRI $x = 0$.)

$$\rightarrow \underbrace{\int_0^L \hat{w} \cdot EA \frac{d^2u(x)}{dx^2} dx}_{\text{PER PARTES}} + \int_0^L \hat{w}\, bA\, dx = 0$$

PER PARTES

$$\int u\,dv = uv - \int v\,du$$

$$\left[\hat{w}\, \underbrace{EA}_{F} \frac{d^2u(x)}{dx}\right]_0^L - \int_0^L \frac{d\hat{w}(x)}{dx} EA \frac{du(x)}{dx} dx$$

$$= \hat{w}\, \overbrace{EA \frac{du(x)}{dx}}^{F}\Big|_{x=L} - \hat{w} EA \frac{du(x)}{dx}\Big|_{x=0} -$$

$$\hat{w}(0) = 0$$

$$- \int_0^L \frac{d\hat{w}(x)}{dx} EA \frac{du(x)}{dx} + \int \hat{w}\, bA\, dx$$

$\hat{w}(x=L) = \hat{w}_L$

$$= \hat{w}_L F - \int_0^L \frac{d\hat{w}(x)}{dx} EA \frac{du(x)}{dx} dx + \int_0^L \hat{w}\, bA\, dx$$

UREDIM:

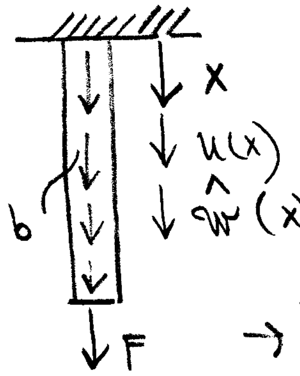
$$\boxed{\int_0^L \frac{d\hat{w}(x)}{dx} EA \frac{du(x)}{dx} = \int_0^L \hat{w}\, bA dx + \hat{w}_L F} \quad (*)$$

ŠIBKA OBLIKA ROBNEGA PROBLEMA

ČE POZNAM A, E, b, F, L IN REŠIM ENAČBO (*) ZA $u(x)$ PRI POLJUBNEM DOPUSTNEM $w(x)$, BO $u(x)$ REŠITEV ŠIBKE OBLIKE ROBNEGA PROBLEMA.

V ELASTIČNOSTI IMA ŠIBKA OBLIKA LEPO FIZIKALNO RAZLAGO, KI JI PRAVIMO PRINCIP VIRTUALNEGA DELA.

ČE SI ZAMISLIMO $\hat{w}$ KOT VIRTUALNI POMIK, POTEM BO REŠ (*) TISTI POMIK $u(x)$ PRI KATEREM BO "KONSTRUKCIJA" V RAVNOTEŽ.

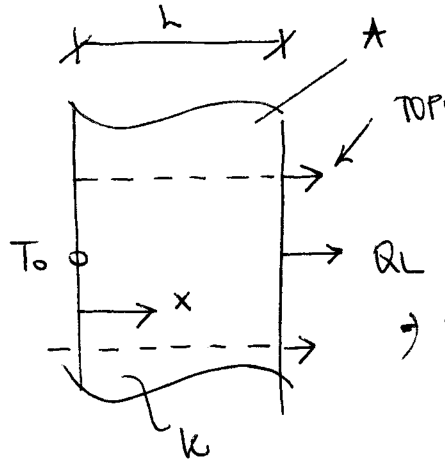


$= \frac{d\hat{w}(x)}{dx}$ (DEFORMACIJA)

→ ČE JE VIRTUALNEMU DELU ZADOŠČENO, JE PALICA V RAVNOTEŽJU.

ŠIBKA OBLIKA: ZMANJŠANE ZAHTEVE (1. ODVOD, PREJ 2), REŠITEV JE PRIBLIŽNA.

C.2. PREVAJANJE TOPLOTE 1D



→ DELČEK STENE

L [m] ... DEBELINA STENE
A [m²] ... POVRŠINA
T_0 [K] ... TEMPERATURA $x = 0$
Q_L [W/m²] ... TOPLOTNI TOK PRI $x = L$
k [W/mK] ... KOEFICIENT TOPLOTNE PREVODNOSTI

$\boxed{DE}$: $kA \cdot \frac{d^2 T(x)}{dx} = 0$

$\boxed{RP}$: $x = 0: \; T(0) = T_0$
$x = L: \; Q(L) = -k \cdot \left.\frac{dT(x)}{dx}\right|_{x=L}$

ROBNI PROBLEM V MOČNI OBLIKI

ŠIBKA OBLIKA:

1.) POMNOŽIMO S TESTNO FUNKCIJO $\hat{w}(x)$

$\hat{w}(x = 0) = 0$

$\hat{w}(x)$ MORA BITI ENAKA 0, TAM KJER JE PREDPISANA $T(x)$.

2.) INTEGRIRAMO PO L:

$$\int_0^L \hat{w}(x) \frac{d^2T}{dx^2} kA\, dx = 0$$

3.) UPORABIMO INTEGRACIJO PER PARTES

$$\hookrightarrow \int_0^L u v' dv = -\int_0^L u' v\, dx + [uv]_0^L$$

$u \rightarrow \hat{w}$, $u' \rightarrow \hat{w}'$

$v' \rightarrow T'' = \frac{d^2T}{dx^2}$, $v \rightarrow T'$

$$0 = \left[\hat{w} \frac{dT(x)}{dx} kA\right]_0^L - \int_0^L \frac{d\hat{w}(x)}{dx} kA \frac{dT(x)}{dx} dx$$

4.) UPOŠTEVAMO ROBNE POGOJE

$x = L$: $Q(L) = -k \cdot \left.\frac{dT}{dx}\right|_{x=L}$

$x = 0$: $\hat{w}(0) = 0$

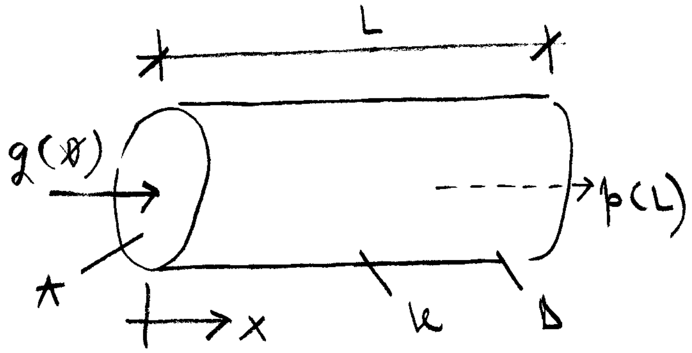
$$\left[\hat{w}(x) \frac{dT(x)}{dx} kA\right]_{x=L} - \overbrace{\left[\hat{w}(x) \frac{dT(x)}{dx} kA\right]_{x=0}}^{0} =$$

$$= \int_0^L \frac{d\hat{w}(x)}{dx} kA \cdot \frac{dT(x)}{dx} dx$$

$$\boxed{-\hat{w}(L)\, Q(L)\, A = \int_0^L \frac{d\hat{w}(x)}{dx} kA \cdot \frac{dT(x)}{dx} dx}$$

$\hookrightarrow$ ŠIBKA OBLIKA ROBNEGA PROBLEMA

C.5. 1D TOK PO CEVI



D [m] ... PREMER CEVI
q [m^3/s] ... PRETOK
p [kN/m^2] ... PRITISK V TEKOČINI
$k = \dfrac{\pi D^4}{128\,\mu}$ ... „TOGOST" TEKOČINE
μ ... VISKOZNOST
A [m^2] ... PREREZ CEVI

$\boxed{\text{DE}}$: $kA \cdot \dfrac{d^2 p(x)}{dx^2} = 0$

$\boxed{\text{RP}}$: $x = 0 : \quad -kA \dfrac{dp(x)}{dx} = q(0)$

$x = L : \quad p(L) = p_L$

$\hat{w}(x)$ MORA BITI 0 PRI $x = L$, KJER JE PREDPISAN $p(x)$.

$$\int_0^L \hat{w}(x)\, kA \frac{d^2 p(x)}{dx^2}\, dx = 0$$

$$0 = \left[\hat{w}\, kA \frac{dp(x)}{dx}\right]_0^L - \int_0^L \frac{d\hat{w}(x)}{dx}\, kA \frac{dp(x)}{dx}\, dx$$

$$\Big[\underbrace{\hat{w}}_{\hat{w}(L)=0}\, kA \frac{dp(x)}{dx}\Big]_{x=L} - \Big[\hat{w}\, \underbrace{kA \frac{dp(x)}{dx}}_{-q(0)}\Big]_{x=0} =$$

$$- \int_0^L \frac{d\hat{w}}{dx}\, kA \frac{dp}{dx}\, dx$$

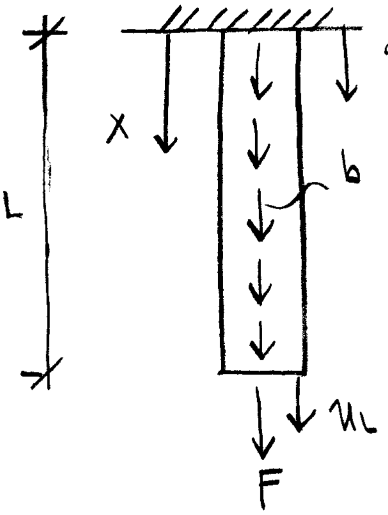
$$\boxed{\hat{w}(0)\, q(0) = \int_0^L \frac{d\hat{w}}{dx}\, kA \frac{dp(x)}{dx}\, dx}$$

↳ ŠIBKA OBLIKA ROBNEGA PROBLEMA

D. REŠEVANJE 1D PROBLEMOV Z GALERKINOVO METODO

D.1. 1D ELASTIČNOST

1.) INTERPOLIRAMO NEZNANO FUNKCIJO $u(x)$ PO OBMOČJU $x \in [0,L]$. NAJ BO INTERPOLACIJA LINEARNA.



$u_0 = 0$ (PODPORA!)

$$u(x) = u_0 \left(1 - \frac{x}{L}\right) + u_L \frac{x}{L}$$

... ČE UPOŠTEVAMO ROBNE POGOJE

$$u(x) = u_L \frac{x}{L} \qquad (u_0 = 0)$$

$\hat{w}(x)$... INTERPOLIRAMO NA ENAK NAČIN

$$\hat{w}(x) = \hat{w}_0 \left(1 - \frac{x}{L}\right) + \hat{w}_L \cdot \frac{x}{L}$$

$\hat{w}_0 = \hat{w}(x=0) = 0$, KER JE PRI $x=0$ PREDPISAN POMIK $u(x=0) = u_0$.

→ VELIKOST POMIKA NI POMEMBNA, POMEMBNO JE SAMO, DA JE POMIK TAM PREDPISAN.

$$\boxed{\hat{w}_L F + \int_0^L \hat{w}(x)\, b A \, dx = \int_0^L \frac{d\hat{w}(x)}{dx} EA \cdot \frac{du(x)}{dx} dx}$$

↳ ŠIBKA OBLIKA ROBNEGA PROBLEMA

VSTAVIMO INTERPOLACIJO V ŠIBKO OBLIKO R.P.:

$$\hat{w}_L F + \int_0^L \underbrace{\hat{w}_L \frac{x}{L}}_{= \hat{w}(x} bA \, dx = \int_0^L \underbrace{\hat{w}_L \frac{1}{L}}_{\frac{d\hat{w}}{dx}} EA \cdot \underbrace{\frac{1}{L} u_L}_{\frac{du}{dx}} dx$$

→ PREUREDIMO PREJŠNJO ENAČBO:

$$\hat{w}_L \left(F + \int_0^L \frac{x}{L} bA\,dx - \int_0^L \frac{EA}{L^2} u_L\,dx \right) = 0$$

$$\hat{w}_L \left(F + \frac{bA}{L} \cdot \frac{x^2}{2}\Big|_0^L - \frac{EA}{L^2} u_L \cdot x\Big|_0^L \right) = 0$$

$$\hat{w}_L \left(F + \frac{LbA}{2} - \frac{EA}{L} u_L \right) = 0$$

$\hat{w}_L$ ↓ POLJUBNA VREDNOST

u_L → ?

DOBIMO 1 ENAČBO ZA 1 NEZNANKO u_L:

$$F + \frac{bAL}{2} = \frac{EA}{L} u_L \rightarrow u_L = \frac{LF}{EA} + \frac{bL^2}{2E}$$

$$\boxed{u(x) = u_L \cdot \frac{x}{L} = \left(\frac{LF}{EA} + \frac{bL^2}{2E} \right) \frac{x}{L} = x\left(\frac{F}{EA} + \frac{bL}{2E} \right)}$$

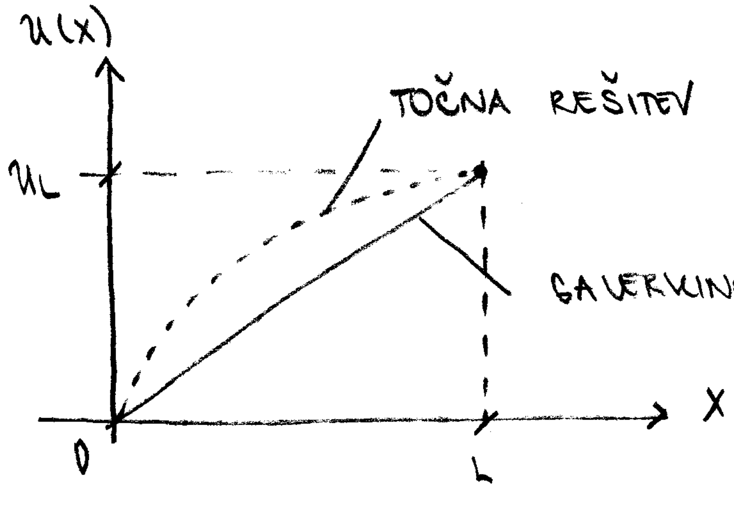
↳ RAČUNSKA REŠITEV DE

↳ LINEARNA OBLIKA ENAČBE

→ TOČNA REŠITEV DE: $\boxed{u(x) = \frac{F}{EA} x + \frac{bL}{E} x - \frac{b}{E} \frac{x^2}{2}}$

↳ KVADRATIČNA OBLIKA ENAČBE

$$x = L: \quad u(L) = \ldots = \frac{FL}{EA} + \frac{bL^2}{2E}$$



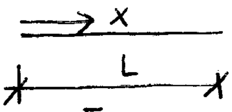
↳ OSNOVA ZA MKE

ŠIBKA OBLIKA JE MANJ STROGA OD DE (MOČNE OBLIKE): DOVOLJUJE, DA JE $\frac{du(x)}{dx} = \text{KONST} \neq 0$, MEDTEM KO DE ZA REŠITEV DOVOLI FUNKCIJE, KI IMAJO $\frac{d^2u}{dx^2} = \text{KONST} \neq 0$.

→ KAJ DELAMO? PONOVITEV!

UKVARJAMO SE Z REŠEVANJEM 1D PROBLEMOV V GRADBENIŠTVU (RAZTEZANJE PALICE, TOK V CEVI, PREVAJANJE TOPLOTE)

↳ ROBNI PROBLEM



└ DIFERENCIALNA ENAČBA (ZA $\forall\, x \in [0,L]$)

└ ROBNI POGOJI PRI $x=0$ IN $x=L$

} $\boxed{\text{ANALITIČNA REŠITEV}}$

↓

ŠIBKA OBLIKA ROBNEGA PROBLEMA

└ PREVEDEMO DE (UPOŠTEVAMO TUDI ROBNE POGOJE) V INTEGRABILNO OBLIKO

↓

PRIBLIŽNO REŠEVANJE Z GALERKINOVO METODO

↓

METODA KONČNIH ELEMENTOV

$\boxed{\text{MKE}}$

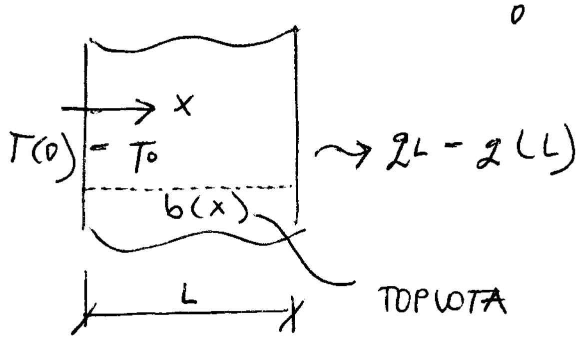
D.2. 1D PREVAJANJE TOPLOTE

ŠIBKA OBLIKA:

$$\int_0^L \frac{d\hat{w}(x)}{dx}\, kA\, \frac{dT(x)}{dx}\, dx = \int_0^L \hat{w}(x)\, b(x)\, A\, dx - \hat{w}(L)\, q_L$$

$\hat{w}$ – VIRTUALNI POMIK

→ $\hat{w}$ MORA VELJATI ZA ∀ FUNKC



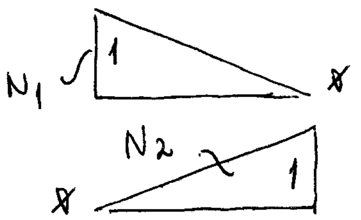
POSTOPEK:

1.) INTERPOLIRAMO $T(x)$ PO VSEM OBMOČJU $x \in [0,L]$, TAKO, DA BO $T(0)=T_0$.

TO NAREDIMO Z LINEARNO KOMBINACIJO DVEH FUNKC

2013/14 1. URŠKA BAČAR NM.

$$T(x) = N_1(x) \cdot T_0 + N_2(x) \underbrace{T_L}_{T(x=L)}$$



IZBEREM SI DVE FUNKCIJI
→ KAKŠNI STA FUNKCIJI N_1 IN N_2?

$$N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

$$T(x) = \left(1 - \frac{x}{L}\right) T_0 + \frac{x}{L} \cdot T_L$$ → TO FUNKCIJO LAHKO UPORABIM ZA APROKSIMACIJO

2.) INTERPOLIRAM $\hat{w}(x)$ Z ENAKIMI FUNKCIJAMI. UPOŠTEVAM, DA JE $\hat{w}(x) = 0$ PRI $x = 0$.

$$\hat{w}(x) = N_1(x) \cdot \underbrace{0}_{\hat{w}(x=0)} + N_2(x) \underbrace{\hat{w}_L}_{\hat{w}(x=L)} = N_2(x)\, \hat{w}_L = \frac{x}{L} \hat{w}_L$$

→ POIŠČEMO VREDNOSTI MED DISKRETNIMI TOČKAMI, VMES INTERPOLIRAMO (LINEAREN POTEK)

FUNKCIJA KI JO POIŠČEMO, ZADOŠČA ŠIBKI OBLIKI.

3.) UPORABIMO INTERPOLACIJI V ŠIBKI OBLIKI

$$\int_0^L \frac{\hat{w}_L}{L} kA \left(\frac{-T_0}{L} + \frac{T_L}{L}\right) dx \underset{\substack{\uparrow \\ b(x)=0}}{=} -\hat{w}_L q_L A$$

4.) PREUREDIM:

$$\hat{w}_L \left[\int_0^L \frac{kA}{L^2}(-T_0 + T_L)\, dx + q_L A\right] = 0$$

KER JE $\hat{w}_L$ POLJUBEN:

$$-\frac{kA}{L} T_0 + \frac{kA}{L} T_L + q_L A = 0 \quad \rightarrow \quad \boxed{T_L = T_0 - \frac{q_L L}{k}}$$

TOREJ: $T(x) = T_0 - \frac{x}{L} T_0 + \frac{x}{L} T_0 - \frac{x q_L}{k} =$

$$T(x) = T_0 - x \frac{q_L}{k}$$

→ PRIMERJAVA Z ANALITIČNO REŠITVIJO:

V TEM PRIMERU JE GALERKINOVA REŠITEV <u>ENAKA</u> ANALITIČNI. ZAKAJ?

KER TUDI DE: KA $\frac{d^2T(x)}{dx^2} = 0$ ZAHTEVA LINEARNO FUNKCIJO, TOREJ ENAKEGA REDA KOT SMO JO UPORABILI PRI INTERPOLACIJI $T(x)$.

D.3. 1D TOK V CEVOVODU

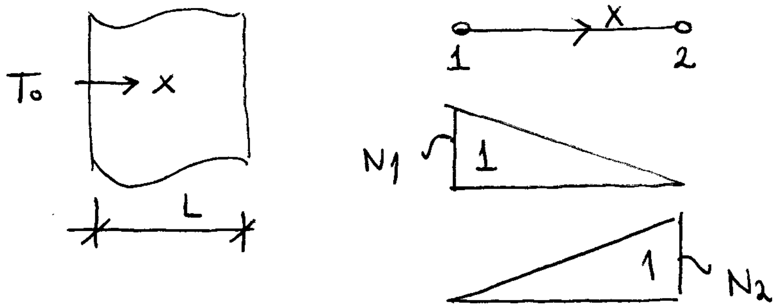
ENAKO KOT PRI D.2. NAREDI SAM!

D.4. GALERKINOVA METODA - ZAKLJUČEK

BISTVO METODE:

- <u>INTERPOLIRAM NEZNANO FUNKCIJO</u> V <u>ŠIBKI OBLIKI</u> Z <u>LINEARNO KOMBINACIJO ZNANIH FUNKCIJ</u> PO VSEM OBMOČJU $x \in [0, L]$.
- INTERPOLACIJA MORA ZADOSTITI <u>ROBNEMU POGOJU</u>.
- INTERPOLIRAM TESTNO FUNKCIJO NA ENAK NAČIN. TESTNA FUNKCIJA MORA BITI 0 TAM, KJER JE TA NEZNANA FUNKCIJA PREDPISANA.

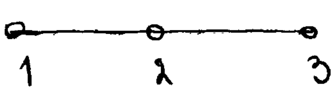
PRIMER: <u>INTERPOLACIJA 1</u>

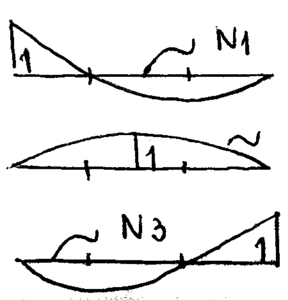


$$T(x) = N_1(x)T_0 + N_2(x)T_L$$
$$\hat{w}(x) = N_2(x)\hat{w}_L$$

DOBIMO IZ ŠIBKE OBLIKE 2 ENAČBI (ZA T_2 IN T_L)

<u>INTERPOLACIJA 2</u>





$$T(x) = N_1(x)T_0 + N_2(x)T_2 + N_3(x)T_L$$
$$\hat{w}(x) = N_2(x)\hat{w}_2 + N_3(x)\hat{w}_L$$

→ V ENAKEM SMISLU LAHKO NADALJUJEMO ŠE NAPREJ.

V 1D JE TO MOGOČE, V 2D IN 3D PA VELIKO TEŽJE NAJDEMO PRIMERNE KANDIDATE (FUNKCIJE).

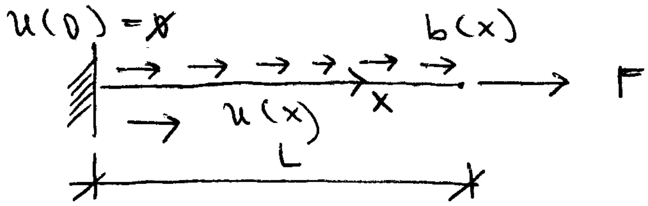
ZATO SO SE RAZVILI KONČNI ELEMENTI, PRI KATERIH TE FUNKCIJE DEFINIRANE "LOKALNO" (LE NA ENEM KE), IN NE PO CELEM OBMOČJU, KJER IŠČEMO REŠITEV.

E. METODA KONČNIH ELEMENTOV

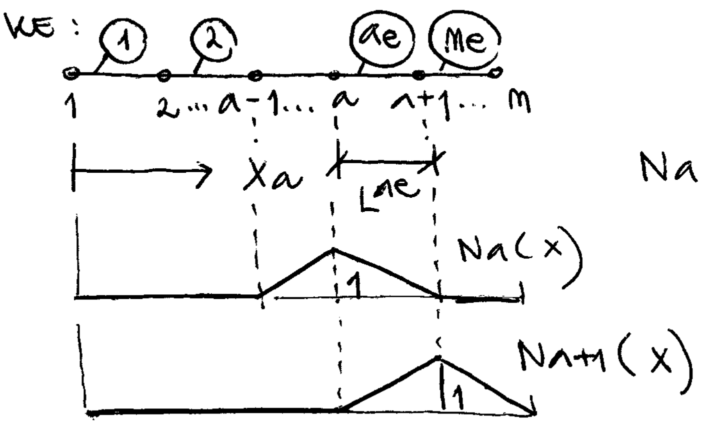
= ZELO PODOBNA KOT GALERKINOVA METODA.

OBMOČJE, PO KATEREM IŠČEMO REŠITEV, RAZDELIMO NA POSAMEZNE DELE (= KONČNE ELEMENTE), NATO PA INTERPOLACIJSKE FUNKCIJE DEFINIRAMO SAMO PO TEM OBMOČJU.

TO JE ENAKO, KOT ČE VZAMEMO GALERKINOVO METODO Z NASLEDNJIMI FUNKCIJAMI:



$$u(x) = \sum_{a=1}^{M_V} N_a(x)\, u_a$$



$$N_a(x) = \begin{cases} \cdot\ \dfrac{x_{a+1} - x_a}{L^{ae}}, & x \in [x_a, x_{a+1}] \\ \cdot\ \dfrac{x - x_{a-1}}{L^{ae-1}}, & x \in [x_{a-1}, x_a] \\ \cdot\ \emptyset, & \text{SICER} \end{cases}$$

$$\frac{dN_a(x)}{dx} = \begin{cases} \cdot\ -\dfrac{1}{L^{ae}}, & x \in [x_a, x_{a+1}] \\ \cdot\ \dfrac{1}{L^{ae-1}}, & x \in [x_{a-1}, x_a] \\ \cdot\ \emptyset, & \text{SICER} \end{cases}$$

INTERPOLACIJA:

$$u(x) = \sum_{a=1}^{mv} N_a(x) \cdot u_a \quad ; \quad \underline{u_1 = u(x=0) = 0}$$

UPOŠTEVAM ŠE

$$\hat{w}(x) = \sum_{a=1}^{mv} N_a(x)\, \hat{w}_a \quad ; \quad \underline{\hat{w}_1 = 0}$$

NESEM INTERPOLACIJO V ŠIBKO OBLIKO:

$$\int_0^L \frac{d\hat{w}(x)}{dx} EA \frac{du(x)}{dx} dx = \int_0^L \hat{w}(x) b(x)\, dA + \hat{w}_{mv} F$$

$$\int_0^L \sum_{a=1}^{mv} \frac{dN_a(x)}{dx} EA \sum_{b=1}^{mv} u_b \frac{dN_b(x)}{dx} =$$

$$- \int_0^L \sum_{a=1}^{mv} \hat{w}_a N_a(x)\, b(x)\, dx + \hat{w}_{mv} F$$

$$\sum_{a=1}^{mv} \hat{w}_a \left[\sum_{b=1}^{mv} \underbrace{\int_0^L \frac{dN_a(x)}{dx} EA \frac{dN_b(x)}{dx} dx}_{K_{ab}}\, u_b - \underbrace{\int_0^L N_a(x)\, b\, x\, dx}_{\bar{f}_a} \right] -$$

$$- \hat{w}_{mv} F = 0$$

$$\rightarrow \boxed{\sum_{a=1}^{mv} \hat{w}_a \left[\sum_{b=1}^{mv} K_{ab} u_b - \bar{f}_a \right] - \hat{w}_{mv} F = 0}$$

→ IMA PREDPISANO VREDNOST $\hat{w}_1 = 0$, SICER POLJUBEN

ZA $\forall a$ DOBIMO 1 ENAČBO. ŠTEVILO ENAČB SE Z mv ZREDUCIRA NA men, KO UPOŠTEVAMO ROBNE POGOJE.

VSAKA ENAČBA IMA OBLIKO: $\boxed{\sum_{b=1}^{mv} K_{ab} u_b = f_a}$

- $f_a = \bar{f}_a$, ZA $a \neq mv$
- $f_a = \bar{f}_a \pm F$, ZA $a = mv$

ENAČBE LAHKO ZAPIŠEMO V MATRIČNI OBLIKI:

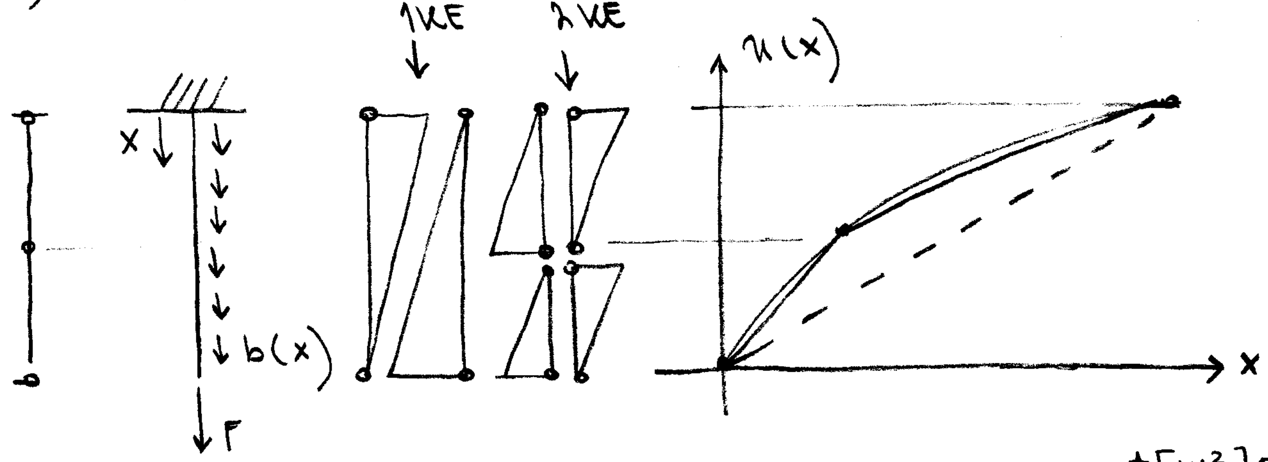
$$\underbrace{\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1\,men} \\ K_{21} & K_{22} & \dots & K_{2\,men} \\ \vdots & & & \\ K_{men\,1} & K_{men\,2} & \dots & K_{men\,men} \end{bmatrix}}_{\text{ČLENI TOGOSTNE MATRIKE}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{en} \end{Bmatrix}}_{\text{POMIKI}} = \underbrace{\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{men} \end{Bmatrix}}_{\text{OBTEŽBA}}$$

- $f_a = \int_0^L N_a \, b(x) \, dx$ ✓
- $f_{Men} = \int_0^L N_{Men} \, b(x) \, dx + F$ ✓

} TO POZNAMO

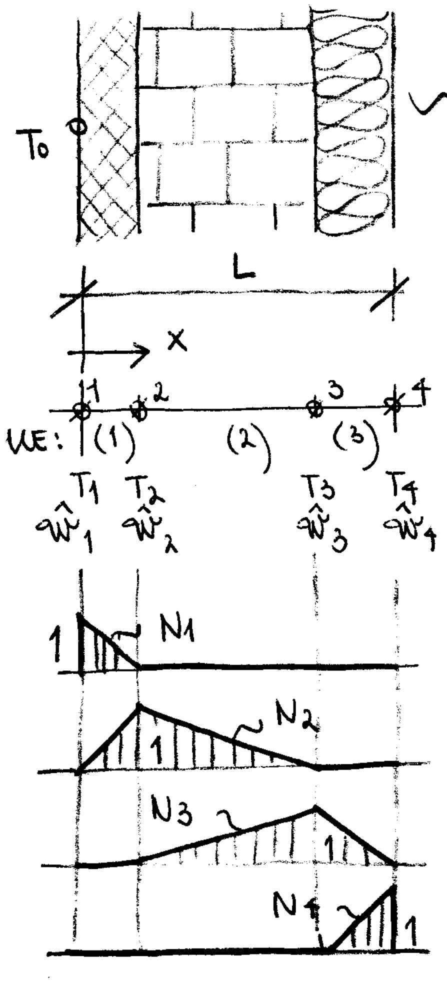
→ IŠČEMO SAMO POMIK

*) PRI PALICI:



E.1. NADALJEVANJE ZGODBE

... 1D PROBLEM: STENA



$\rightsquigarrow q_L$

TESTNA F.

$A[m^2] = 1m^2$

$$\begin{cases} \int_0^L \frac{d\hat{w}(x)}{dx} KA \frac{dT(x)}{dx} dx = -q_L \hat{w}_L \cdot A \\ T(x=0) = T_0 \rightarrow T_1 = T_0 \\ \hat{w}(x=0) = 0 \rightarrow \hat{w}_1 = 0 \end{cases}$$

INTERPOLACIJSKE FUNKCIJE:

$$\to \hat{w}(x) = \sum_{a=1}^{4} N_a(x) \, \hat{w}_a$$

$$\to T(x) = \sum_{a=1}^{4} N_a(x) \, T_a$$

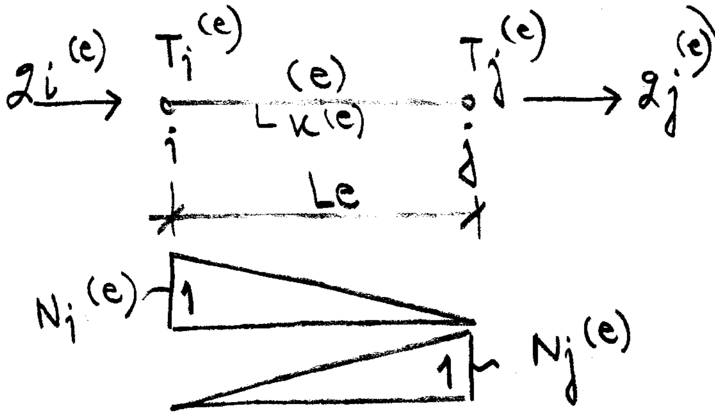
DOBIMO SISTEM LINEARNIH E

$$\underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ \vdots & \vdots & \vdots & \vdots \\ K_{41} & K_{42} & K_{43} & K_{44}\end{bmatrix}}_{\text{TOGOSTNA MATRIKA K.}} \underbrace{\begin{Bmatrix} T_1 = T_0 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}}_{\substack{\text{VEKTOR} \\ \text{VOZLIŠČNIH} \\ \text{TEMPERATUR}}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ -q_L A \end{bmatrix}}_{\substack{\text{OBTEŽNI} \\ \text{VEKTOR} \\ \text{KONSTRUKCIJE}}} \rightarrow \text{IZRAČUNA } T_2, T_3, T_4!$$

$$K_{ab} = \int_0^L \frac{dN_a}{dx} kA \frac{dN_b}{dx} dx$$

E.2. STARŠEVSKI KONČNI ELEMENT

→ ALI OBSTAJA TAKŠEN "STARŠEVSKI" ELEMENT, KI BI LAHKO "DAL" VSE KE V NEKI MREŽI KE?



→ ODGOVOR: DA.

STARŠEVSKI KE

POIŠČEM ŠE ENAČBO STARŠEVSKEGA KE:

→ UPORABIM PRINCIP VIRTUALNEGA DELA (PVD) NA TEM ELEMENTU:

TEMPERATURA PO ELEMENTU

$$\int_0^{L^{(e)}} \frac{d\hat{w}_e(x^{(e)})}{dx^{(e)}} k^{(e)} A^{(e)} \frac{dT^{(e)}(x^{(e)})}{dx^{(e)}} dx^{(e)} =$$

$$= A^{(e)} \underset{\substack{\uparrow \\ \text{GRE V ELEMENT,} \\ \text{ZATO } \oplus}}{q_i^{(e)}} \hat{w}_i^{(e)} - \underset{\substack{\uparrow \\ \text{GRE IZ ELEMENTA,} \\ \text{ZATO } \ominus}}{q_j^{(e)}} \hat{w}_j^{(e)} \cdot A^{(e)}$$

ČE INTERPOLIRAM: $\hat{w}^{(e)}(x^{(e)}) = N_i^{(e)} \hat{w}_i^{(e)} + N_j^{(e)} \hat{w}_j^{(e)}$

IN $T^{(e)}(x^{(e)}) = N_i^{(e)} T_i^{(e)} + N_j^{(e)} T_j^{(e)}$

→ INTERPOLACIJI NESEM V PVD, DOBIM ENAČBO ZA STARŠEVSKI KE!

KAKŠNI STA $N_i^{(e)}$ IN $N_j^{(e)}$?

$$N_i^{(e)}(x^{(e)}) = 1 - \frac{x^{(e)}}{L^{(e)}}$$

$$N_j^{(e)}(x^{(e)}) = \frac{x^{(e)}}{L^{(e)}}$$

SPLOŠNI ČLEN: $k_{ij}^{(e)} = \int_0^{L^{(e)}} \frac{dN_i^{(e)}}{dx^{(e)}} k^{(e)} A^{(e)} \frac{dN_j^{(e)}}{dx^{(e)}} dx^{(e)}$

→ IZ PVD DOBIM:

$$\underbrace{\begin{bmatrix} \frac{k^{(e)}A^{(e)}}{L^{(e)}} & -\frac{k^{(e)}A^{(e)}}{L^{(e)}} \\ -\frac{k^{(e)}A^{(e)}}{L^{(e)}} & \frac{k^{(e)}A^{(e)}}{L^{(e)}} \end{bmatrix} \begin{Bmatrix} T_i^{(e)} \\ \vdots \\ T_j^{(e)} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} q_i^{(e)}A^{(e)} \\ \vdots \\ -q_j^{(e)}A^{(e)} \\ \vdots \end{Bmatrix}}_{\text{ENAČBA "RAVNOTEŽJA" STARŠEVSKEGA KE}}$$

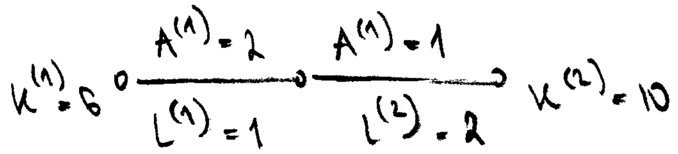
VPLIV SOSEDNJIH KE

TEMPERATURA V VOZLIŠČIH

$$i \quad \int_0^{L^{(e)}} \frac{1}{L^{(e)}} \frac{1}{L^{(e)}} k^{(e)} A^{(e)} dx^{(e)}$$

TOGOSTNA MATRIKA KE (ZA 1D PREVAJANJE TOPLOTE)

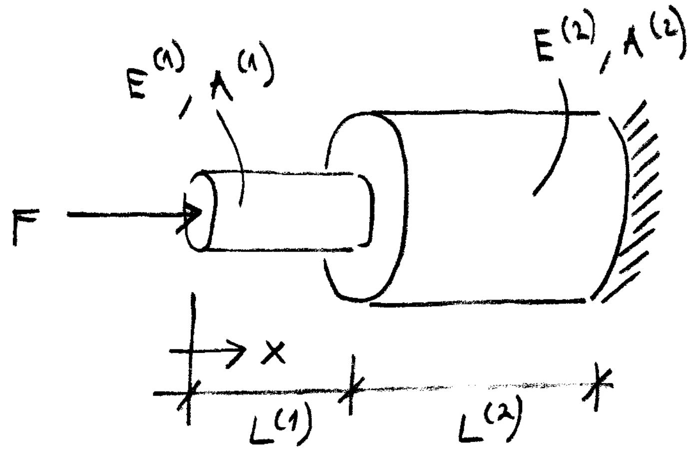
PRIMER:



E.3. KAKO POTEKA RAČUN PO MKE

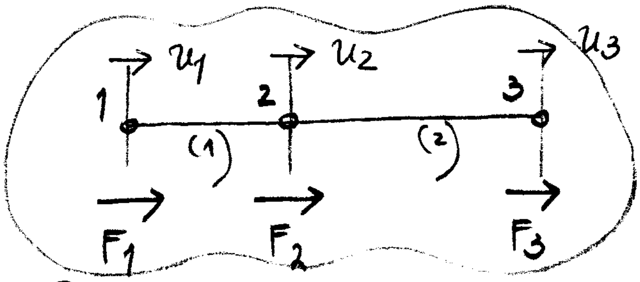
1.) DOLOČI MREŽO KE
2.) DOLOČI OBTEŽNI VEKTOR KONSTRUKCIJE
3.) NAPIŠEM TOGOSTNO MATRIKO ZA VSAK KE IZ MREŽE (UPORABIM STARŠEVSKI KE ZA OBRAVNAVANI PROBLEM)
   └ MORA BITI NA VOLJO
4.) ZDRUŽIM VSE TOGOSTNE MATRIKE KE V TOGOSTNO M. KONST.
5. IZRAČUNAM NEZNANE VREDNOSTI V VOZLIŠČIH MREŽE KE
6. ZRA M ŠE KOLIČINE PO SAKEM KE

E. 4. PRIMER 1



→ IZRAČUNAJ POMIK IN SILE V PALICI!

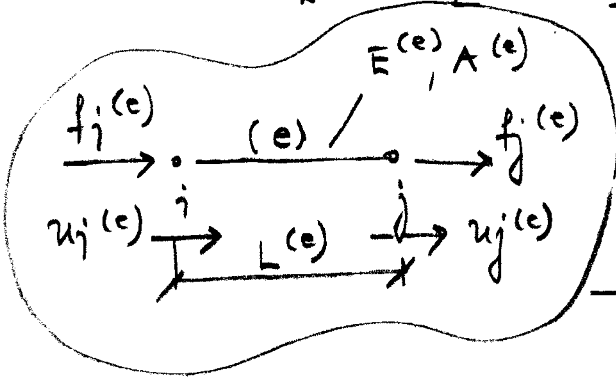
MREŽA KE:



$$\underset{\sim}{F} = \begin{Bmatrix} F_1 = F \\ F_2 = 0 \\ F_3 = \text{REAKCIJA} = ? \end{Bmatrix}$$

PVD ⊕ INTERPOLACIJA

$$\underset{\sim}{K}^{(e)} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} = \frac{E^{(e)} A^{(e)}}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i^{(e)} \\ u_j^{(e)} \end{Bmatrix}$$

ELEMENT 1

$$\begin{Bmatrix} f_i^{(1)} \\ f_j^{(1)} \end{Bmatrix} = \frac{E^{(1)} A^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i^{(1)} \\ u_j^{(1)} \end{Bmatrix}$$

ELEMENT 2

$$\begin{Bmatrix} f_i^{(2)} \\ f_j^{(2)} \end{Bmatrix} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i^{(2)} \\ u_j^{(2)} \end{Bmatrix}$$

ENAČBO KONSTRUKCIJE DOBIM, KO UGOTOVIM NASLEDNJE:

1.) $u_i^{(1)} = u_1$ $\quad u_i^{(2)} = u_2$ $\quad \Rightarrow$

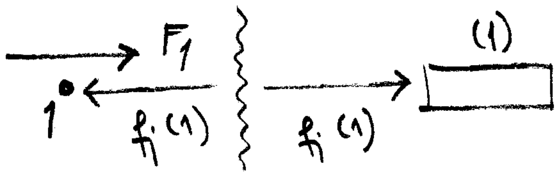
$u_j^{(1)} = u_2$ $\quad u_j^{(2)} = u_3$

$$\begin{Bmatrix} f_i^{(1)} \\ f_j^{(1)} \end{Bmatrix} = \frac{E^{(1)} A^{(1)}}{L^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f_i^{(2)} \\ f_j^{(2)} \end{Bmatrix} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

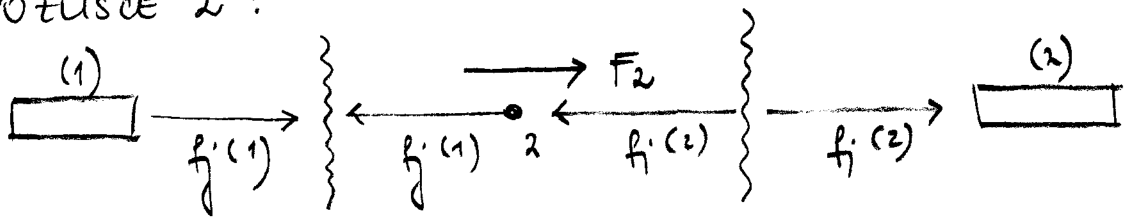
2.) ENAČBO KONSTRUKCIJE DOBIMO, ČE ZAGOTOVIMO RAVNOTEŽJE V VSEH 3 VOZLIŠČIH MREŽE KE

VOZLIŠČE 1:



$$F_1 = f_i^{(1)} = \frac{E^{(1)} A^{(1)}}{L^{(1)}} (u_1 - u_2)$$

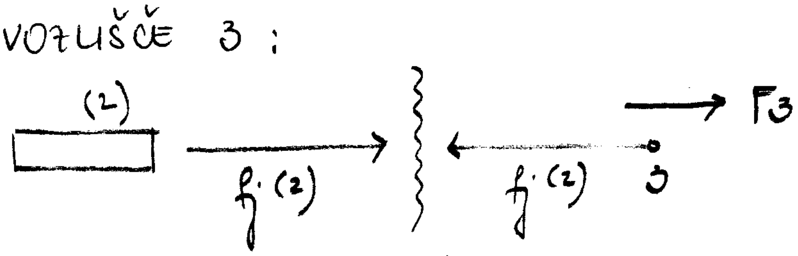
VOZLIŠČE 2:



$$F_2 = f_j^{(1)} + f_i^{(2)}$$

$$F_2 = \frac{E^{(1)} A^{(1)}}{L^{(1)}} (-u_1 + u_2) + \frac{E^{(2)} A^{(2)}}{L^{(2)}} (u_2 - u_3)$$

VOZLIŠČE 3:



$$F_3 = f_j^{(2)} = \frac{E^{(2)} A^{(2)}}{L^{(2)}} (-u_2 + u_3)$$

→ ENAČBE SESTAVIM V MATRIČNO OBLIKO ...

$$\underbrace{\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} \frac{E^{(1)}A^{(1)}}{L^{(1)}} & -\frac{E^{(1)}A^{(1)}}{L^{(1)}} & \emptyset \\ -\frac{E^{(1)}A^{(1)}}{L^{(1)}} & \frac{E^{(1)}A^{(1)}}{L^{(1)}} + \frac{E^{(2)}A^{(2)}}{L^{(2)}} & -\frac{E^{(2)}A^{(2)}}{L^{(2)}} \\ \emptyset & -\frac{E^{(2)}A^{(2)}}{L^{(2)}} & \frac{E^{(2)}A^{(2)}}{L^{(2)}} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}$$

ENAČBA KONSTRUKCIJE

→ NAPIŠEMO V DOBLJENO ENAČBO ROBNE POGOJE:

VEMO: $F_1 = F$, $F_2 = \emptyset$, $u_3 = \emptyset$

OZNAKE:

$$C^{(1)} = \frac{E^{(1)}A^{(1)}}{L^{(1)}}$$

$$C^{(2)} = \frac{E^{(2)}A^{(2)}}{L^{(2)}}$$

$$\begin{Bmatrix} F \\ \emptyset \\ F_3 \end{Bmatrix} = \begin{bmatrix} C^{(1)} & -C^{(1)} & \emptyset \\ -C^{(1)} & C^{(1)} + C^{(2)} & -C^{(2)} \\ \emptyset & -C^{(2)} & C^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \emptyset \end{Bmatrix}$$

? ? ?

→ NAJPREJ IZRAČUNAMO POMIKE:

$$\begin{Bmatrix} F \\ \emptyset \end{Bmatrix} = \begin{bmatrix} C^{(1)} & -C^{(1)} \\ -C^{(1)} & C^{(1)} + C^{(2)} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

- $-C^{(1)} u_1 + (C^{(1)} + C^{(2)}) u_2 = \emptyset$

$$u_1 = \frac{u_2 (C^{(1)} + C^{(2)})}{C^{(1)}} = u_2 \left(1 + \frac{C^{(2)}}{C^{(1)}}\right)$$

- $F = \left(C^{(1)}\left(1 + \frac{C^{(2)}}{C^{(1)}}\right) - C^{(1)}\right) u_2 = C^{(2)} \cdot u_2$

→ $\underline{u_2 = F \frac{1}{C^{(2)}}}$ → $\underline{u_1 = F (C^{(1)} + C^{(2)})}$

→ NATO IZRAČUNAM REAKCIJO

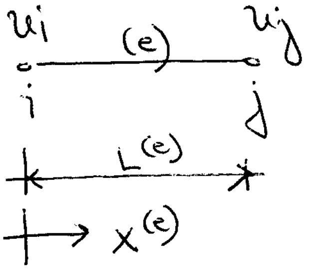
$$F_3 = -c^{(2)} u_2 = -F \checkmark$$

KO IMAMO POMIKE V VOZLIŠČIH MREŽE KE, LAHKO IZRAČUNAMO NOTRANJO SILO V VSAKI PALICI.

TO LAHKO NAREDIMO NA 2 NAČINA:

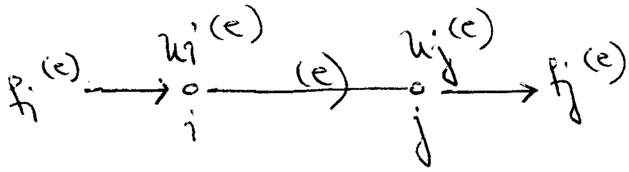
(1) S KONSTITUTIVNO ENAČBO:

$$N^{(e)}(x) = E^{(e)}\,{}^{(e)}\,\frac{du^{(e)}}{dx^{(e)}} = \frac{E^{(e)}A^{(e)}}{L^{(e)}}\left(u_j^{(e)} - u_i^{(e)}\right)$$

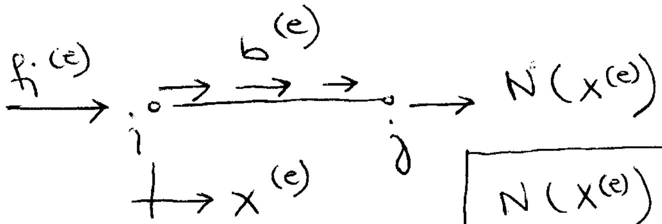


(2) UPORABIM ENAČBO KE

$$\begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} = \frac{E^{(e)}A^{(e)}}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i^{(e)} \\ u_j^{(e)} \end{Bmatrix}$$



→ ČE KE OBTEŽIM ŠE Z ZUNANJO OBREMENITVIJO ⇒



$$\boxed{N(x^{(e)}) = -f_i^{(e)} - b^{(e)} x^{(e)}}$$

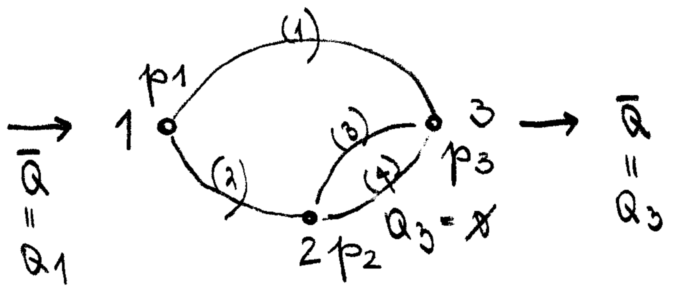
↳ TO LAHKO NAREDIMO SAMO PRI LINIJSKIH KONST. (PRI PLOSKOVNIH NE MOREMO)

$$\begin{Bmatrix} q_i \\ -q_j \end{Bmatrix} = \frac{k^{(e)}}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} p_i \\ p_j \end{Bmatrix}$$

ČE GRE TOK V ELEMENT, JE q_i POZITIVEN ⊕
ČE GRE TOK IZ ELEMENTA, JE q_j NEGATIVEN ⊖

(2) = MOŽEN SAMO ZA 1D PROBLEME + NOSILCE.
ZA 2D, 3D PROBLEME, PLOŠČE, LUPINE, STENE, ...
TO NI MOGOČE.

• MREŽA KE:



• "OBTEŽNI" VEKTOR (VEKTOR PRETOKOV) = "SILA"

$$\underset{\sim}{Q} = \begin{Bmatrix} \bar{Q} \\ \emptyset \\ -\bar{Q} \end{Bmatrix} \begin{matrix} \text{GRE V, } \oplus \\ \\ \text{GRE IZ, } \ominus \end{matrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

• TOGOSTNA MATRIKA "KONSTRUKCIJE"

$$\underset{\sim}{Q} = \underset{\sim}{K} \cdot \underset{\sim}{p}$$

"POMIKI" = p JE VEKTOR PRITISKOV

$\underset{\sim}{K}$ ↓ IŠČEMO

POIŠČEMO "TOGOSTNO MATRIKO KONSTRUKCIJE" $\underset{\sim}{K}$

$\underset{\sim}{K}$ IMA DIMENZIJE (3x3)

$$\underset{\sim}{K} = \bigcup_{e=1}^{4} k^{(e)}$$

(e)	i	j
1	1	3
2	1	2
3	2	3
4	2	3

$$\rightarrow c^{(e)} = \frac{k^{(e)}}{L^{(e)}}$$

$$\underset{\sim}{K}^{(1)} = C^{(1)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

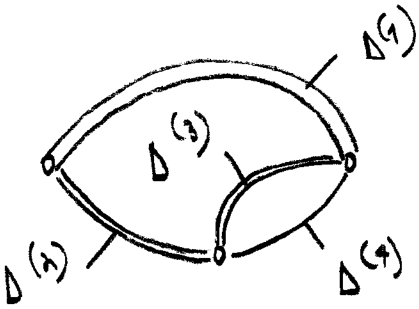
$$\underset{\sim}{K} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|c|c} C^{(1)}+C^{(2)} & -C^{(2)} & -C^{(1)} \\ \hline -C^{(2)} & C^{(2)}+C^{(3)}+C^{(4)} & -C^{(3)}-C^{(4)} \\ \hline -C^{(1)} & -C^{(3)}-C^{(4)} & C^{(4)}+C^{(3)}+C^{(1)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

NADALJUJEMO S ŠTEVILKAMI

$\bar{Q} = 0{,}10 \ m^3/s$

$\mu = 0{,}96 \times 10^{-3} \ Ns/m^2$

CEV	Δ [cm]	L [m]
1	2,50	30
2	2	20
3	2	25
4	1,25	20



$$\left.\begin{aligned} C^{(1)} &= 3{,}3 \times 10^{-7} \\ C^{(2)} &= 2{,}05 \times 10^{-7} \\ C^{(3)} &= 1{,}64 \times 10^{-7} \\ C^{(4)} &= 0{,}39 \times 10^{-7} \end{aligned}\right\} \quad C^{(e)} = \frac{\pi \Delta^{(e)4}}{128 \mu L^{(e)}}$$

$$K = 10^{-7} \begin{bmatrix} 5{,}35 & -2{,}05 & 3{,}3 \\ -2{,}05 & 4{,}00 & -1{,}95 \\ -3{,}3 & -1{,}95 & 5{,}25 \end{bmatrix} \leftarrow$$

TOGOSTNA MATRIKA KONSTRUKCIJE
- SIMETRIČNA
- $\det \underset{\sim}{K} = 0$
  - SINGULARNA

$$\underset{\sim}{Q} = \underset{\sim}{K} \cdot \underset{\sim}{p} \Rightarrow \begin{Bmatrix} 0{,}10 \\ 0 \\ -0{,}10 \end{Bmatrix} = \underset{\sim}{K} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \end{Bmatrix}$$

→ NIČ NE MOREMO IZRAČUNATI, KER JE MATRIKA SINGULARNA. ZATO PREDPIŠEMO EN PRITISK V VOZLIŠČU IN IZRAČUNAMO OSTALE VOZLIŠČNE PRITISKE RELATIVNO GLEDE NA PREDPISANEGA. NAJ BO $p_3 = 0$.

2013/14 1. UROŠKA BAČAR NM.

→ DOBIM REDUCIRAN SISTEM ENAČB, KI GA LAHKO REŠIM:

$$\begin{Bmatrix} 0'10 \\ 0 \end{Bmatrix} = 10^{-7} \begin{bmatrix} 5'35 & -2'05 \\ -2'05 & 4'00 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

$p_1 = 0'232 \times 10^6 \ N/m^2$

$p_2 = 0'119 \times 10^6 \ N/m^2$

$$q^{(e)} = -k^{(e)} \frac{dp^{(e)}}{dx^{(e)}} = -k^{(e)} \frac{p_j^{(e)} - p_i^{(e)}}{L^{(e)}}$$

PRETOKI SO:

$q^{(1)} = 0'077 \ m^3/s$

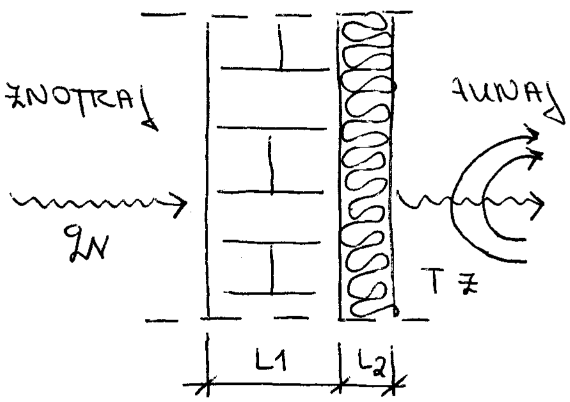
$q^{(2)} = 0'023 \ m^3/s$

$q^{(3)} = 0'019 \ m^3/s$

$q^{(4)} = 0'0039 \ m^3/s$

E.G. PRIMER 3

PREREZ STENE:



A ... PREREZ [m^2]

T_a ... TEMPERATURA AMBIENTA (ZRAKA)

$\beta (T_Z - T_a)$

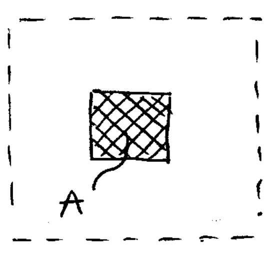
↳ KOEFICIENT TOPLOTNE PRESTOPNOSTI

[W/m^2K]

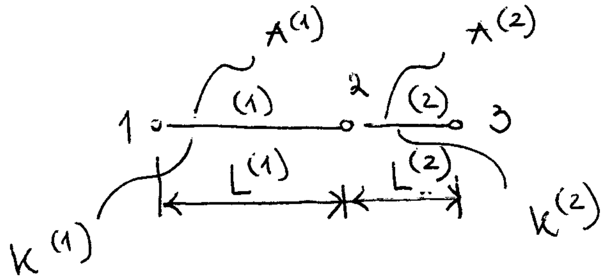
$\frac{1}{\beta}$ [m^2K/W]

↳ KOEFICIENT TOPLOTNE UPORNOSTI

POGLED:

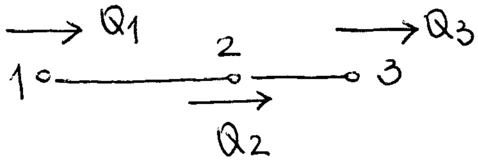


<u>NALOGA</u>: IZRAČUNAJ T IN TOPLOTNI TOK PO PREREZU STENE Z MKE:



$$A^{(1)} = A^{(2)} = A$$

↳ KOEFICIENT TOPLOTNE PREVODNOSTI



GRE NOTER ⇒ ⊕

OBTEŽNI VEKTOR $\underset{\sim}{Q} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} q_N \cdot A \\ 0 \\ -\beta\,(T_3 - T_a) \end{Bmatrix}$

GRE VEN ⇒ ⊖

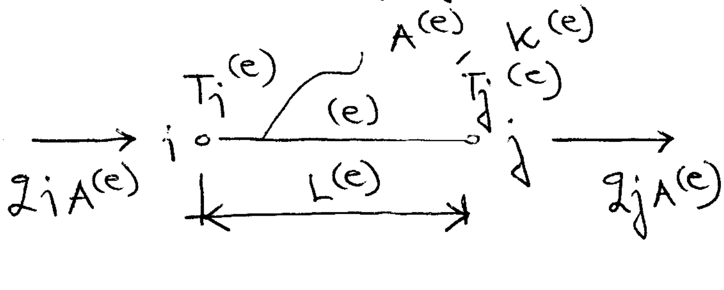
VEKTOR TEMPERATUR $\underset{\sim}{T} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$

ENAČBA ZA MREŽO KE: $\underset{\sim}{Q} = \underset{\sim}{K} \cdot \underset{\sim}{T}$

"TOGOSTNA" MATRIKA KONSTRUKCIJE

$\underset{\sim}{K}$ JE SESTAVLJEN IZ "TOGOSTNIH" MATRIK VSEH KE IZ MREŽE.

POTREBUJEMO ENAČBO ZA TIPIČNI KE (e), KI BO OPISOVALA PREVAJANJE TOPLOTE PO TEM KE.



$$\boxed{qA = -k\,\frac{dT}{dx}} \longrightarrow q_i A^{(e)} = -k^{(e)} A^{(e)}\,\frac{T_j^{(e)} - T_i^{(e)}}{L^{(e)}}$$

$$q_j A^{(e)} = -k^{(e)} A^{(e)}\,\frac{T_j^{(e)} - T_i^{(e)}}{L^{(e)}}$$

2013/14 1. URŠKA BAČAR NM.

DOBLJENA $\underset{\sim}{k}$ NI SINGULARNA.

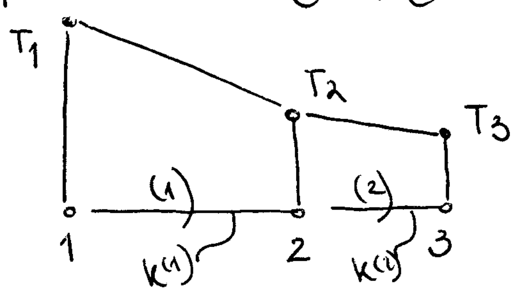
(GLEJ ANALOGIJO S PALICO Z VZMETJO)

ZATO NI POTREBNO $\underset{\sim}{k}$ REDUCIRATI. IZRAČUNAM VSE 3 TEMPERATURE:

$$T_3 = T_a + \frac{2N}{p}$$

$$T_2 = T_3 + \frac{A\,2N}{C^{(2)}}$$

$$T_1 = T_3 + \frac{A\,(C^{(1)} + C^{(2)})}{C^{(1)} + C^{(2)}}$$



ČE HOČEMO IZRAČUNATI TOPLOTNI TOK PO KE (e), UPORABIMO

$q^{(e)} = -k^{(e)} \cdot \frac{T_j^{(e)} - T_i^{(e)}}{L^{(e)}}$, KI PRIDE IZ $q = -k\frac{dT}{dx}$.

$(e) = 1, 2$

$T_i^{(1)} = T_1$, $T_i^{(2)} = T_2$

$T_j^{(1)} = T_2$, $T_j^{(2)} = T_3$

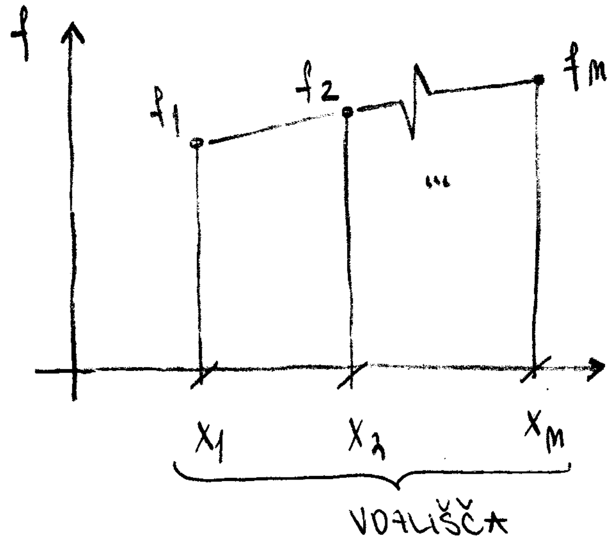
2. DEL : NUMERIČNI POSTOPKI, KI JIH POTREBUJEMO PRI MKE

A. INTERPOLACIJA

PO KE JE POTREBNO INTERPOLIRATI FUNKCIJE, KI NAS ZANIMAJO (NPR. POMIKI, TEMPERATURE, PRITISK)

A.1. LAGRANGEVA INTERPOLACIJA

UPORABIMO JO, ČE POZNAMO VREDNOST FUNKCIJ V DOLOČENIH TOČKAH. IMAMO TABELO (f_1, x_1), (f_2, x_2), (f_3, x_3), ... (f_m, x_m).

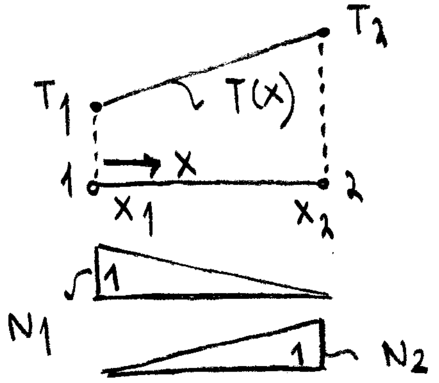


IDEJA:

POMNOŽI f_i Z $N_i(x)$, KI IMA VREDNOST 1 V x_i IN VREDNOST 0 V VSEH OSTALIH VOZLIŠČIH. DOBLJENE FUNKCIJE SEŠTEJ.

→ DOBIŠ LAGRANGEVO INTEGRACIJO ČEZ m VOZLIŠČ

PRIMER 1: KE Z 2 VOZLIŠČI

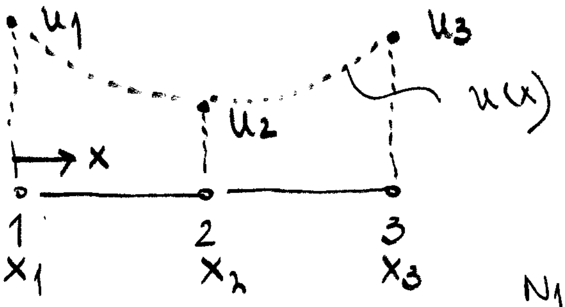


$$T(x) = T_1 N_1(x) + T_2 N_2(x)$$

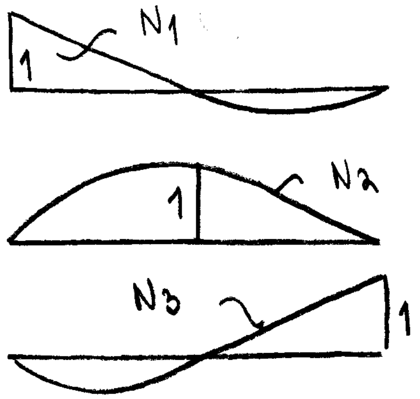
$$N_1(x) = \begin{cases} 1, & x = 0 \\ 0, & x = L \end{cases}$$

$$N_2(x) = \begin{cases} 1, & x = L \\ 0, & x = 0 \end{cases}$$

PRIMER 2: KE S 3 VOZLIŠČI (NI V SAP-u)



$$u(x) = u_1 N_1(x) + u_2 N_2(x) + u_3 N_3(x)$$



$$N_1(x) = \frac{(x-x_2)}{(x_1-x_2)} \cdot \frac{(x-x_3)}{(x_1-x_3)}$$

$$N_2(x) = \frac{(x-x_1)}{(x_2-x_1)} \cdot \frac{(x-x_3)}{(x_2-x_3)}$$

$$N_3(x) = \frac{(x-x_1)}{(x_3-x_1)} \cdot \frac{(x-x_2)}{(x_3-x_2)}$$

POSPLOŠITEV NA m TOČK:

$$u(x) = u_1 N_1(x) + u_2 N_2(x) + \underbrace{\ldots}_{u_i N_i(x)} + u_m N_m(x)$$

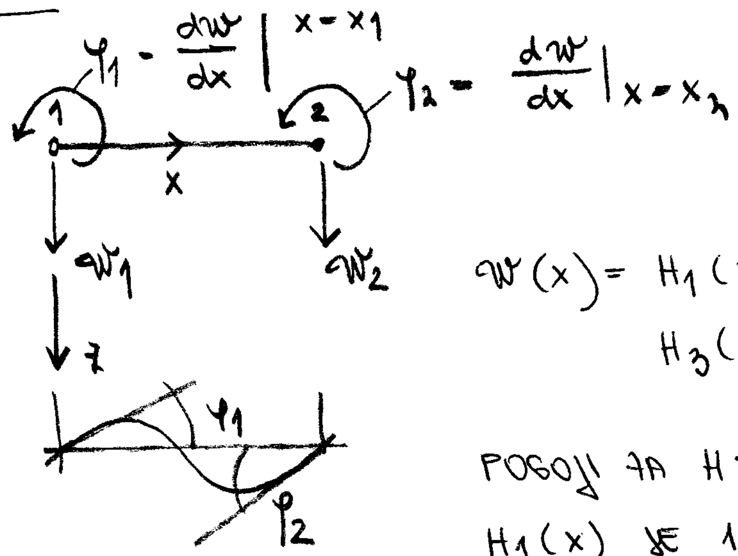
$$N_i(x) = \frac{(x-x_1)}{(x_i-x_1)} \cdot \frac{(x-x_2)}{(x_i-x_2)} \cdots \frac{(x-x_{i-1})}{(x_i-x_{i-1})} \frac{(x-x_{i+1})}{(x_i-x_{i+1})} \cdots \frac{(x-x_m)}{(x_i-x_m)} = \begin{cases} 1, & \text{PRI } x = x_i \\ 0, & \text{PRI } \forall x_j \neq x_i \end{cases}$$

TO SO LAGRANGEVI POLINOMI

2.2. HERMITOVA INTERPOLACIJA

TO INTERPOLACIJO UPORABIMO, ČE IMAMO V POSAMEZNIH TOČKAH ZNANE TAKO VREDNOSTI FUNKCIJE, KOT TUDI NJENIH ODVODOV.

PRIMER : 2 VOZLIŠČNI KE ZA NOSILCE



$$w(x) = H_1(x)\,w_1 + H_2(x)\,\varphi_1 + H_3(x)\,w_2 + H_4(x)\,\varphi_2$$

POGOJI ZA $H_i(x)$:

$H_1(x)$ JE 1 PRI $x = x_1$
$H_1'(x)$ JE 0 PRI $x = x_1$
$H_1(x)$ JE 0 PRI $x = x_2$
$H_1'(x)$ JE 1 PRI $x = x_2$

$$\frac{dw}{dx} = H_1'(x)\,w_1 + H_2'(x)\,\varphi_1 + H_3'(x)\,w_2 + H_4'(x)\,\varphi_2$$

$H_2(x)$ JE 0 PRI $x = x_1$
$H_2'(x)$ JE 1 PRI $x = x_1$
$H_2(x)$ JE 0 PRI $x = x_2$
$H_2'(x)$ JE 0 PRI $x = x_2$

$H_3(x)$ JE 1 PRI $x = x_2$
$H_3'(x)$ JE 0 PRI $x = x_2$
$H_3(x)$ JE 0 PRI $x = x_1$
$H_3'(x)$ JE 0 PRI $x = x_1$

$H_4(x)$ JE 0 PRI $x = x_2$
$H_4'(x)$ JE 1 PRI $x = x_2$
$H_4(x)$ JE 0 PRI $x = x_1$
$H_4'(x)$ JE 0 PRI $x = x_1$

VSAK $H_i(x)$ MORA IZPOLNITI 4 POGOJE. TOREJ JE LAHKO KUBIČNA FUNKCIJA.

$$\boxed{H_i(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3}$$

KONSTANTE c_0, c_1, c_2, c_3 IZRAČUNAMO IZ TEH 4 POGOJEV. ČE TO NAREDIMO, DOBIMO HERMITOVE POLINOME:

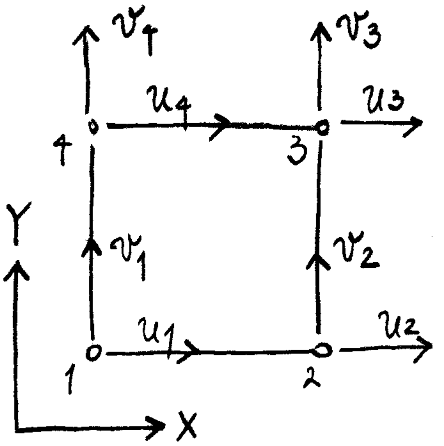
$$\boxed{\begin{aligned} H_1 &= 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1 \\ H_2 &= \frac{x^3}{L^2} - 2\frac{x^2}{L} + x \\ H_3 &= -2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2 \\ H_4 &= \frac{x^3}{L^2} - \frac{x^2}{L} \end{aligned}}$$

$L = x_2 - x_1, \quad x_2 > x_1$

A.3. LAGRANGEVA INTERPOLACIJA PO PRAVOKOTNIKU

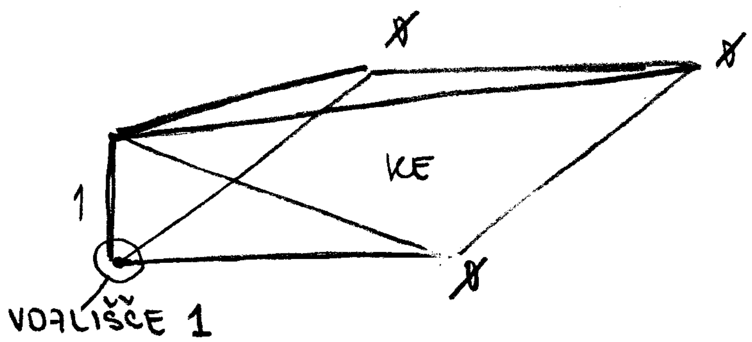
INTERPOLIRAMO POMIK PO 4-VOZLIŠČNEM KE. POMIK PO KE OPIŠEMO S FUNKCIJO:

$$u(x,y) = N_1(x,y)\,u_1 + N_2(x,y)\,u_2 + N_3(x,y)\,u_3 + N_4(x,y)\,u_4$$



$N_i(x,y)$... LAGRANGEVE INTERPOLACIJSKE FUNKCIJE

$$N_i(x,y) = \begin{cases} 1 & \text{V VOZLIŠČU } i \\ 0 & \text{V VOZLIŠČIH } j,\ j \neq i \end{cases}$$



$$N_1(x,y) = \frac{1}{(-L_x)(-L_y)}(x - x_2)(y - y_3)$$

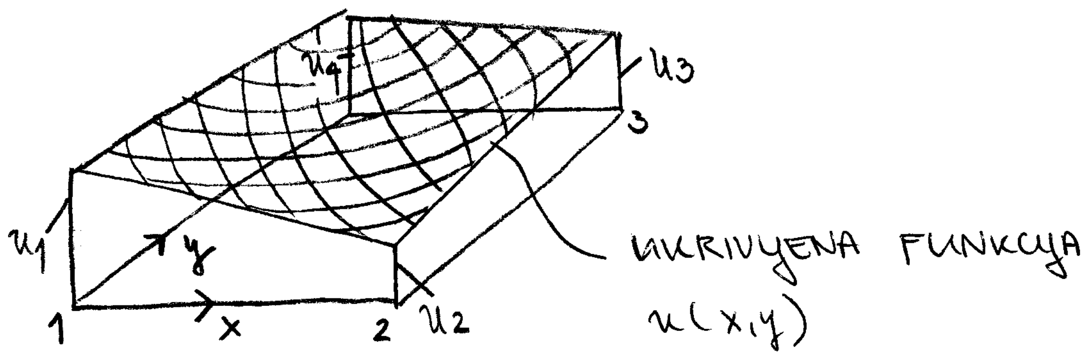
1.) $N_1(x_1, y_1) = \frac{1}{(\cancel{-L_x})(\cancel{-L_y})} \underbrace{(x_1 - x_2)}_{\cancel{L_x}} \underbrace{(y_1 - y_3)}_{\cancel{L_y}} = 1$

2.) $N_2(x_2, y_2) = \frac{1}{L_x \, (-L_y)} (x - x_1)(y - y_3)$

3.) $N_3(x_3, y_3) = \frac{1}{L_x \cdot L_y} (x - x_1)(y - y_3)$

4.) $N_4(x_4, y_4) = \frac{1}{(-L_x) L_y} (x - x_1)(y - y_3)$

$$v(x,y) = N_1(x,y) v_1 + N_2(x,y) v_2 + N_3(x,y) v_3 + N_4(x,y) v_4$$

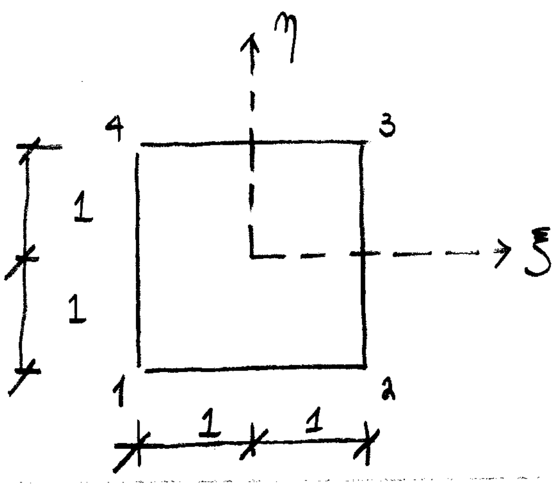


→ POMIKA u IN v STA LINEARNA PO x IN PO y PO PRAVOKOTNEM KONČNEM ELEMENTU.

A.4. LAGRANGEVA INTERPOLACIJA PO KVADRATU

KVADRAT S STRANICO 2.

KS: ξ, η V TEŽIŠČU KE.



LAGRANGEVE INTERPOLACIJSKE FUNKCIJE ZA TA KVADRAT:

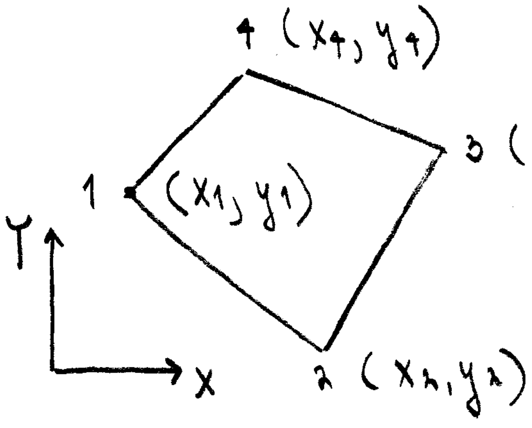
$$N_i(\xi,\eta) = \begin{cases} 1 & , \text{V VOZLIŠČU } i \\ 0 & , \text{V VOZLIŠČU } j,\ i \neq j \end{cases}$$

$$N_1(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta-1)$$

$$\left.\begin{array}{l} \xi_1 = -1 \\ \eta_1 = -1 \end{array}\right\} \quad N_1(\xi_1,\eta_1) = \frac{1}{4}(-1-1)(-1-1) = 1$$

$$N_2(\xi,\eta) = -\frac{1}{4}(\xi+1)(\eta-1)$$
$$N_3(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)$$
$$N_4(\xi,\eta) = -\frac{1}{4}(\xi-1)(\eta+1)$$

A.5. LAGRANGEVA INTERPOLACIJA PO POLJUBNEM 4-KOTNIKU



→ VZAMEMO POLJUBNI ŠTIRI-VOZLIŠČNI KE.

LEŽI V RAVNINI XY.

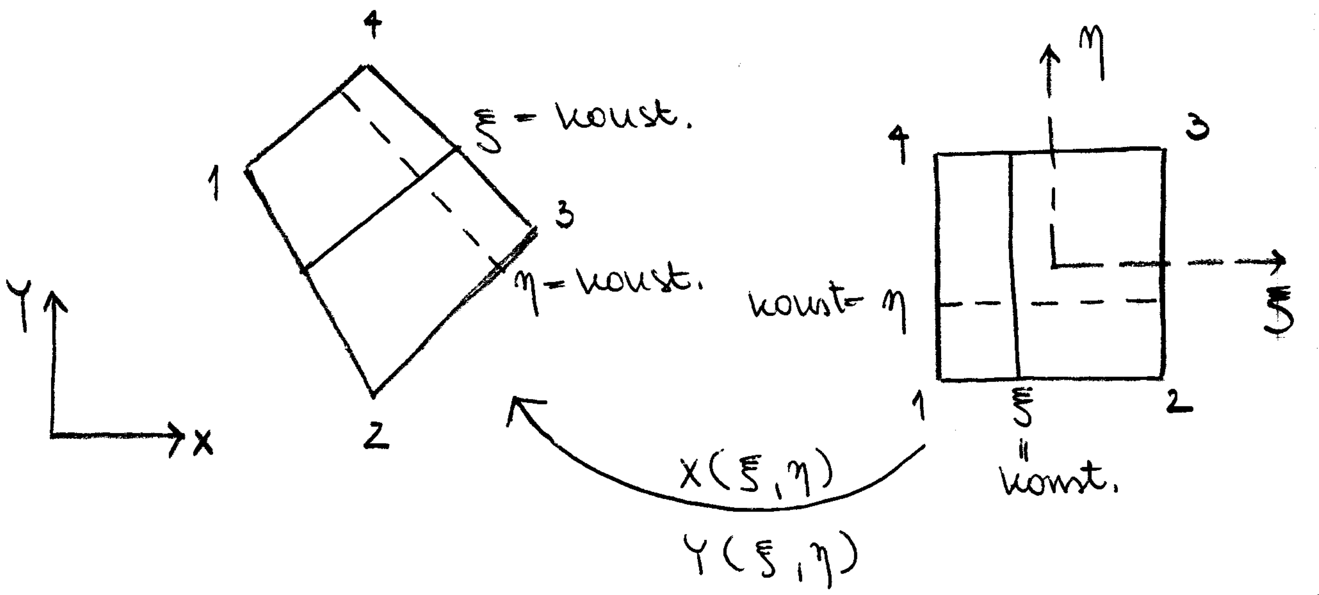
POSKUSIMO PONOVNO DOLOČITI INTERPOLACIJSKE FUNKCIJE, KI JIH JE ZA TAK PRIMER TEŽKO DOLOČITI.

$$N_i(x,y) = \begin{cases} 1 & , \text{V VOZLIŠČU } i \\ 0 & , \text{V VOZLIŠČU } j,\ j \neq i \end{cases}$$

POSTOPEK POENOSTAVIMO: POLJUBEN 4-KOTNIK PRESLIKA V KVADRAT S STRANICO 2 IN V KS ξ, η (KER SMO S TAKIM KE DELALI V TOČKI A.4.)

PRESLIKAVI: $X(\xi,\eta)$, $Y(\xi,\eta)$

TA PRESLIKAVA OMOGOČA, DA DOBIMO SLIKO $u(x,y)$ IZ POMIKA $u(\xi,\eta)$.



PRESLIKAVI $X(\xi,\eta)$, $Y(\xi,\eta)$ DOBIMO TAKO, DA REČEMO:

NAJ BO PRI $\xi = -1$, $\eta = -1$ JE x_1, y_1
PRI $\xi = 1$, $\eta = 1$ JE x_2, y_2
PRI $\xi = -1$, $\eta = -1$ JE x_3, y_3
PRI $\xi = 1$, $\eta = 1$ JE x_4, y_4

$$X(\xi,\eta) = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)x_4$$

$$Y(\xi,\eta) = N_1(\xi,\eta)y_1 + N_2(\xi,\eta)y_2 + N_3(\xi,\eta)y_3 + N_4(\xi,\eta)y_4$$

ZA VSAKO TOČKO ξ,η DOBIMO SLIKO xy.

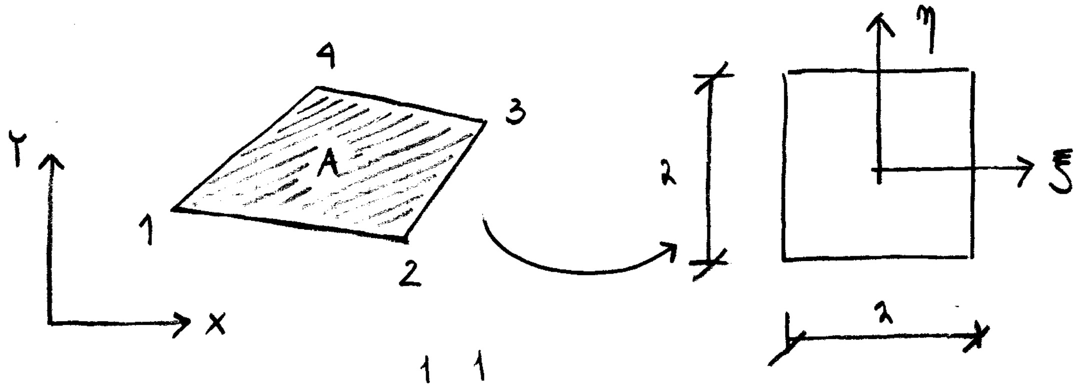
LAHKO SI TUDI PREDSTAVLJAMO DA STA ξ IN η PARAMETRA, KI OPIŠETA POVRŠINO KE.

ZA RAZLIČNE VREDNOSTI ξ = konst. DOBIMO ENO DRUŽINO ČRT, PODOBNO ZA RAZLIČNE VREDNOSTI η = konst. DOBIMO DRUGO DRUŽINO ČRT.

JAKOBIJAN PRESLIKAVE IZ OBMOČJA V xy-RAVNINI V OBMOČJE V $\xi\eta$-RAVNINI JE $\rightarrow$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_i \frac{\partial N_i}{\partial \xi} x_i & \sum_i \frac{\partial N_i}{\partial \eta} x_i \\ \sum_i \frac{\partial N_i}{\partial \xi} y_i & \sum_i \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

POGLEJMO SI ŠE, KAKO INTEGRIRAMO PO POLJUBNEM 4-KOTNIKU, ČE IMAMO OBMOČJE A KE V RAVNINI xy.



$$\int_A f(x,y)\, dA = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta)) \det \underset{\sim}{J}\, d\xi\, d\eta$$

TOREJ LAHKO TUDI INTEGRALE PO POLJUBNEM 4-KOTNIKU IZVEDEMO TAKO, DA INTEGRIRAMO PO KVADRATU S STRANICO 2.

B. NUMERIČNE INTEGRACIJE

KDAJ IN ZAKAJ UPORABITI NUMERIČNO INTEGRACIJO?

- ČE ŽELIMO IZVREDNOTITI INTEGRAL, KI NIMA ANALITIČNE REŠITVE
- ČE JE IZVREDNOTENJE NUMERIČNE INTEGRACIJE HITREJŠE KOT IZVREDNOTENJE ANALITIČNEGA INTEGRALA.

PRI MKE NUM. INTEGRACIJO UPORABLJAMO ZA IZRAČUN TOGOSTNIH MATRIK KE.

ZA VEČINO KE LAHKO TOGOSTNO MATRIKO ZAPIŠEMO LE V INTEGRALSKI OBLIKI.

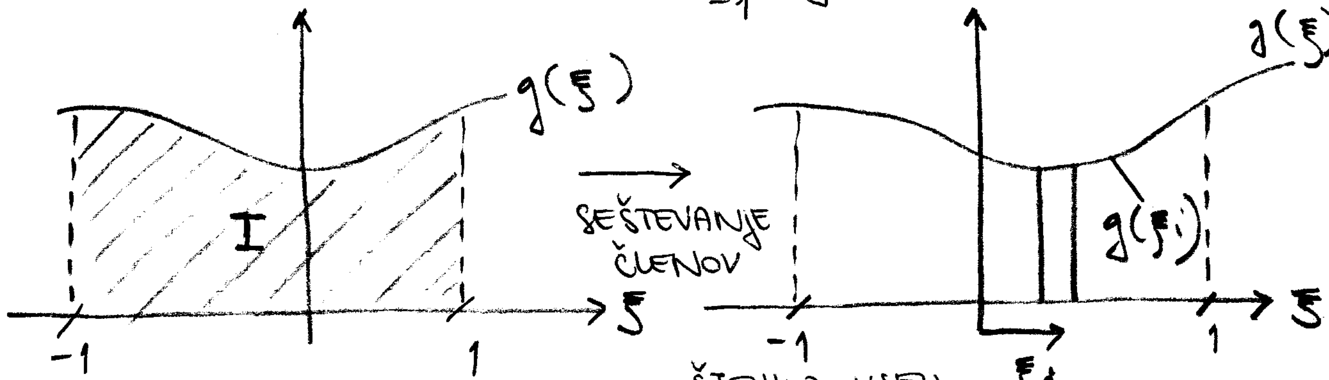
ZA ∀ KE V MREŽI KE MORAMO POSEBAJ IZVREDNOTITI INTEGRAL ZA IZRAČUN TOGOSTNE MATRIKE.

IZJEMA: KE ZA PALICE, KE ZA NOSILCE
↳ IMAMO ŽE PRIPRAVLJENE ANALITIČNE OBLIKE

8.1. GAUSSOVA NUMERIČNA INTEGRACIJA

OSNOVNA IDEJA TE NUM. INTEGRACIJE JE, DA SE INTEGRAL POLJUBNEGA POLINOMA PREVEDE NA SEŠTEVANJE ČLENOV.

$$I = \int_{-1}^{1} g(\xi)\, d\xi$$



N_g – ŠTEVILO VSEH GAUSSOVIH INTEGRACIJSKIH

$$I = \int_{-1}^{1} g(\xi)\, d\xi = \sum_{i=1}^{N_g} w_i\, g(\xi_i)$$

POLJUBNI POLINOM

TOČNO ENAKO!

UTEŽ i-TE TOČKE

VNAPREJ PREDPISANA VREDNOST KOOR. ξ = LOKACIJA GAUSSOVE INTEGRACIJSKE TOČK

DEFINICIJSKO OBMOČJE POLINOMA : $\mathcal{D}_f = [-1, 1]$

w_i = VNAPREJ DOLOČENA VREDNOST UTEŽI, KI PRIPADA i-TI GAUSSOVI INTEGRACIJSKI TOČKI.

LOKACIJE INTEGRACIJSKIH TOČK ξ_i IN VREDNOSTI UTEŽI w_i SO PRI GAUSSOVI NUMERIČNI INTEGRACIJI FIKSNE.

DOLOČIMO JIH TAKO, DA VELJA TOČNA ENAKOST MED ANALITIČNIM IN NUMERIČNIM INTEGRALOM.

8.2. PRIMER

LINEARNA FUNKCIJA : $g(\xi) = c_0 + c_1 \xi$

INTEGRAL REŠIMO ANALITIČNO:

$$I = \int_{-1}^{1} (c_0 + c_1 \xi)\, d\xi = c_0 \xi \Big|_{-1}^{1} + c_1 \frac{\xi^2}{2} \Big|_{-1}^{1} =$$

$$= c_0 (1 - (-1)) + c_1 \left(\frac{1}{2} - \frac{1}{2}\right) = \underline{\underline{2c_0}}$$

INTEGRAL REŠIMO NUMERIČNO:

$$I = w_1\, g(\xi_1) = w_1 (c_0 + c_1 \xi) = w_1 c_0 + w_1 c_1 \xi_1$$

POSKUSIMO, ČE ZADOSTUJE 1 SAMA INTEGRACIJSKA TOČKA

ČE IZENAČIMO ANALITIČNO IN GAUSSOVO INTEGRACIJO:

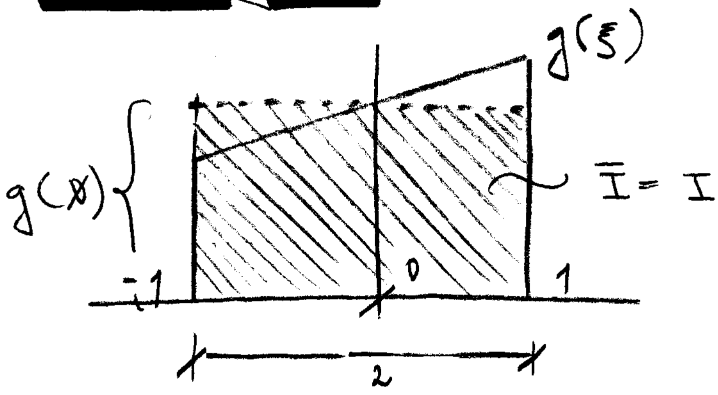
$$2c_0 + 0\, c_1 = w_1 c_0 + w_1 c_1 \xi_1$$

SLEDI : $w_1 \xi_1 = 0 \longrightarrow \xi_1 = 0$

$w_1 = 2$

INTEGRAL POLINOMA 1. REDA $g(\xi)$ NA OBMOČJU $\xi \in [-1, 1]$ LAHKO TOČNO IZRAČUNAMO KOT

$$\boxed{\bar{I} = 2 \cdot g(0)}$$



→ DODATEN PRIMER: $g(\xi) = 2\xi + 3$

POLINOM 1. REDA

$$I = \int_{-1}^{1} g(\xi)\, d\xi \underset{\uparrow}{=} 2 \cdot g(0) = 2 \cdot 3 = \underline{\underline{6}}$$

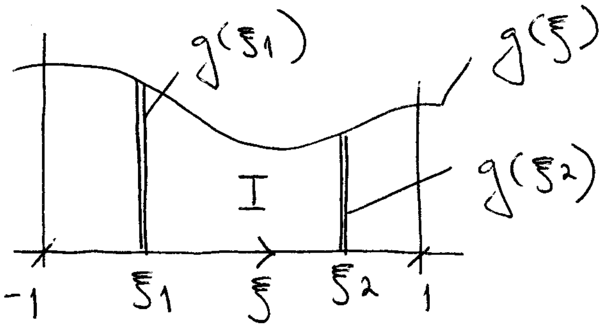
UPORABIMO 1 - TOČKOVNO GAUSSOVO INTEGRACIJO

PONOVITEV: GAUSSOVA NUMERIČNA INTEGRACIJA

IDEJA: $I = \int_{-1}^{1} g(\xi)\, d\xi \underset{\uparrow}{=} \sum_{i=1}^{N_g} g(\xi_i)\, w_i$

TOČNO

DVOTOČKOVNI INTEGRAL:



B.3. POLINOM $g(\xi)$ JE TRETJEGA REDA

IMAM: $g(\xi) = C_0 + C_1\xi + C_2\xi^2 + C_3\xi^3$

VPRAŠANJE: KOLIKO INTEGRACIJSKIH TOČK RABIM, DA BO GAUSSOVA NUM. INTEGRACIJA TOČNA ZA TAK POLINOM?

$N_g = ?$

KJE SO KOORDINATE TEH TOČK?

KOLIKŠNE SO UTEŽI?

2013/14 1. URŠKA BAČAR NM.

$$I = \int_{-1}^{1} g(\xi)\,d\xi = \sum_{i=1}^{Ng} w_i\, g(\xi_i)$$

TOČNO

?

KAJ NAREDIM?

(Ng = 1 BI BILO VERJETNO PREMALO)

POSKUSIM, ČE JE Ng = 2 V REDU.

INTEGRAL IZVREDNOTIM:

$$I = \underbrace{\int_{-1}^{1} g(\xi)\,d\xi}_{\text{ANALITIČNO (A)}} = \underbrace{\sum_{i=1}^{Ng} w_i\, g(\xi_i)}_{\text{NUMERIČNO (N)}}$$

$$A = \left[c_0\xi + \frac{1}{2}c_1\xi^2 + \frac{1}{3}c_2\xi^3 + \frac{1}{4}c_3\xi^4 \right]_{-1}^{1} =$$

$$= 2c_0 + \cancel{0}\,c_1 + \frac{2}{3}c_2 + \cancel{0}\,c_3$$

$$B = w_1(c_0 + c_1\xi_1 + c_2\xi_1^2 + c_3\xi_1^3) +$$
$$w_2(c_0 + c_1\xi_2 + c_2\xi_2^2 + c_3\xi_2^3)$$
$$= c_0(w_1 + w_2) + c_1(w_1\xi_1 + w_2\xi_2) +$$
$$+ c_2(w_1\xi_1^2 + w_2\xi_2^2) + c_3(w_1\xi_1^3 + w_2\xi_2^3)$$

KER A = B, SLEDI:

$$\underbrace{\begin{array}{ll} w_1 + w_2 = 2 & w_1\xi_1^2 + w_2\xi_2^2 = \frac{2}{3} \\ w_1\xi_1 + w_2\xi_2 = \cancel{0} & w_1\xi_1^3 + w_2\xi_2^3 = \cancel{0} \end{array}}$$

V MATHEMATICI (SOLVER), POIŠČEM w_1, w_2, ξ_1, ξ_2

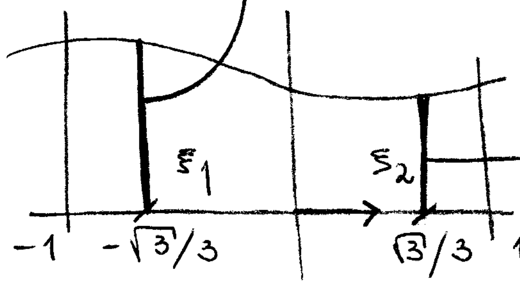
↓

$$w_1 = w_2 = 1 \qquad \xi_1 = -\frac{\sqrt{3}}{3} \qquad \xi_2 = \frac{\sqrt{3}}{3}$$

→ PRIMER : $g(\xi) = 2\xi^3 + 6\xi - 2$

$$\int_{-1}^{1} g(\xi)\,d\xi = \int_{-1}^{1} (2\xi^3 + 6\xi - 2)\,d\xi =$$

$$= \underbrace{1 \cdot \left(2 \cdot (-\sqrt{3}/3)^3 + 6 \cdot (-\sqrt{3}/3) - 2\right)} +$$



$$+ \underbrace{1\left(2(\sqrt{3}/3)^2 + 6(\sqrt{3}/3) - 2\right)}$$

$$= -4$$

B.4. POSPLOŠITEV

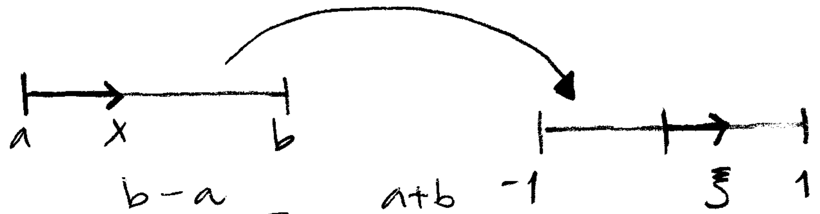
POLINOM 1. REDA ... TOČNA INTEGRACIJA Z ① INTEGRACIJSKO TOČKO , $N_g = 1$

POLINOM 3 . REDA ... –II– Z ② INTEGRACIJSKIMA TOČKAMA $N_g = 2$

ŠT. INTEGRACIJSKIH TOČK

N_g	TOČNA INTEGRACIJA ZA POLINOM $2N_g - 1$
1	1 (LINEAREN POLINOM)
2	3 (KUBIČEN POLINOM)
3	5 (POLINOM 5. REDA)
⋮	
N_g	$(2N_g - 1)$

→ MEJE INTEGRALA : $\int_a^b g(x)\,dx = \int_{-1}^{1} g(\xi)\,\frac{b-a}{2}\,d\xi$



$$x = \frac{b-a}{2}\xi + \frac{a+b}{2}$$

JAKOBIJAN

↳ $dx = J\,d\xi \quad \left(J = \frac{b-a}{2}\right)$

→ VSAK INTEGRAL LAHKO PREVEDEMO NA $\int_{-1}^{1} \dots d\xi$.

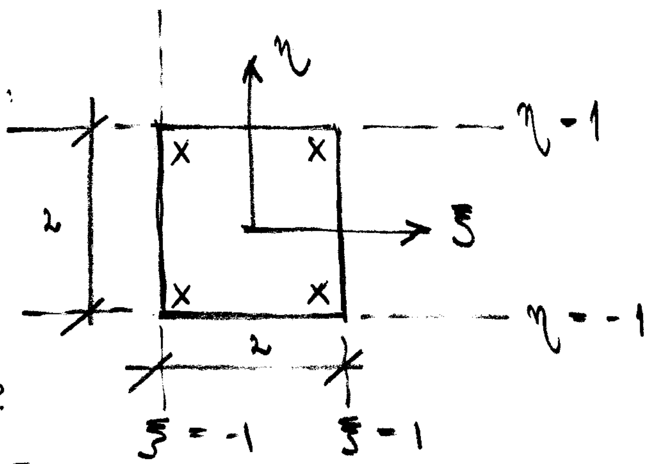
<u>PRIMER</u>: $\int_{2=a}^{3=b} x^2 dx = \int_{-1}^{1} \left(\frac{1}{2}\xi + \frac{5}{2}\right)^2 \underbrace{\frac{3-2}{2}}_{J} d\xi = *$

N_g	w_i	ξ_i
1	$w_1 = 2$	$\xi_1 = 0$
2	$w_1 = w_2 = 1$	$\xi_1 = -\sqrt{3}/3$, $\xi_2 = \sqrt{3}/3$
3	$w_1 = 5/9$ $w_2 = 8/9$ $w_3 = 5/9$	$\xi_1 = -\sqrt{3}/3$, $\xi_2 = 0$ $\xi_3 = \sqrt{3/5}$

$$* = 1\left(\frac{1}{2}\left(-\frac{\sqrt{3}}{3}\right) + \frac{5}{2}\right)^2 \frac{1}{2} + 1 \cdot \left(\frac{1}{2}\left(\frac{\sqrt{3}}{3}\right) + \frac{5}{2}\right)^2 \frac{1}{2} = \underline{\underline{\frac{28}{36}}}$$

8.5. GAUSSOVA NUMERIČNA INTEGRACIJA V 2D

INTEGRIRAMO PO KVADRATU:



$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\, d\xi\, d\eta$$

- NUMERIČNA INTEGRACIJA?

INTEGRIRAJMO NAJPREJ PO ξ, POTEM PO η

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\, d\xi\, d\eta = \int_{-1}^{1} \left(\sum_{i=1}^{N_{g\xi}} g(\xi_i,\eta)\, w_i\right)$$

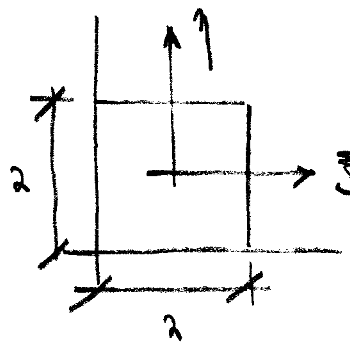
$$= \sum_{j=1}^{N_{g\eta}} \left(\sum_{i=1}^{N_{g\xi}} g(\xi_i,\eta_j)\, w_i\right) w_j$$

PRAVILO: $\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta =$

$= \sum_{i=1}^{Ng\xi} \sum_{j=1}^{Ng\eta} w_j w_i\, g(\xi_i,\eta_j)$

→ PRIMER: TOČEN INTEGRAL: $\int_{-1}^{1}\int_{-1}^{1} 1\,d\xi\,d\eta = *$

LINEAREN



$1\times 1 = \underline{1}$ { 1 INTEGRACIJSKA T. V ξ; 1 INTEGRACIJSKA T. V η }

$* = \sum_{i=1}^{1}\sum_{j=1}^{1} 1\cdot w_i w_j =$

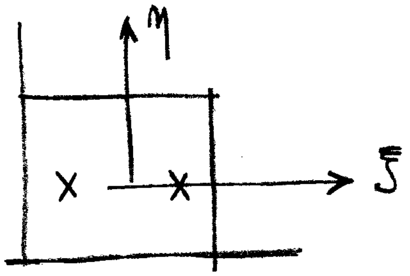
$= 1\cdot 2\cdot 2 = \underline{\underline{4}}$

ZA 1 INTEGRACIJSKO TOČKO: $w_1 = 2$, ... $\xi_1 = 0$

→ PRIMER: $\int_{-1}^{1}\int_{-1}^{1} \xi^2\eta\,d\xi\,d\eta = \sum_{i=1}^{2}\sum_{j=1}^{1} w_i w_j \xi_i^2 \eta_j$

$= \underset{1}{w_1}\,\underset{2}{w_1}\,\xi_1^2\eta_1 + \underset{1}{w_1}\,\underset{2}{w_2}\,\xi_1^2\eta_2 =$

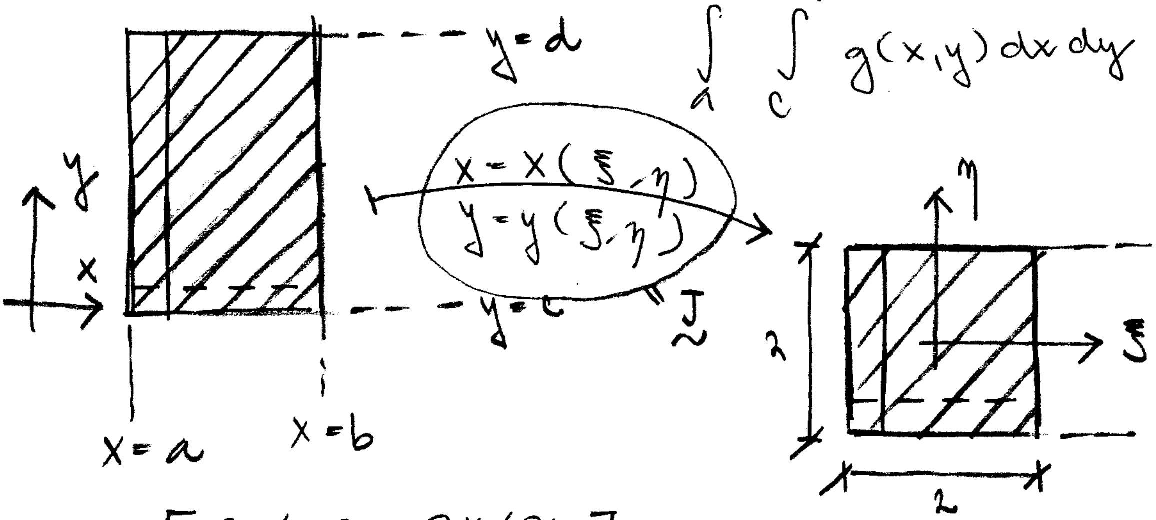
$= 1\cdot 2\cdot(-\sqrt{3}/3)^2\cdot 0 + 1\cdot 2\cdot(\sqrt{3}/3)^2\cdot 0 = \underline{\underline{0}}$



$Ng\xi = 2$, $Ng\eta = 1$ } $2\times 1 = 2$

B.6. GAUSSOVA NUMERIČNA INTEGRACIJA PO PRAVOKOTNIKU

$$\int_a^b \int_c^d g(x,y)\,dx\,dy$$



$$\underset{\sim}{J} = \begin{bmatrix} \partial x/\partial \xi & \partial x/\partial \eta \\ \partial y/\partial \xi & \partial y/\partial \eta \end{bmatrix}$$

$$x(\xi,\eta) = \frac{b-a}{2}\xi + \frac{b+a}{2}$$

$$y(\xi,\eta) = \frac{d-c}{2}\eta + \frac{d+c}{2}$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{b-a}{2} & \cancel{0} \\ \cancel{0} & \frac{b-c}{2} \end{bmatrix} \Rightarrow \det \underset{\sim}{J} = \frac{(b-a)(d-c)}{4}$$

$$\int_a^b \int_c^d g(x,y)\,dx\,dy = \int_{-1}^1 \int_{-1}^1 g(x(\xi,\eta)\ y(\xi,\eta)) \det \underset{\sim}{J}\, d\xi\, d\eta$$

$$= \int_{-1}^1 \int_{-1}^1 f(\xi,\eta)\, d\xi\, d\eta = \sum_{i=1}^{Ng\xi} \sum_{j=1}^{Ng\eta} w_i w_j f(\xi_i, \eta_j)$$

→ <u>PRIMER</u>: $\int_c^d dx\,dy = \int_{-1}^1 \int_{-1}^1 \det J\, d\xi\, d\eta =$

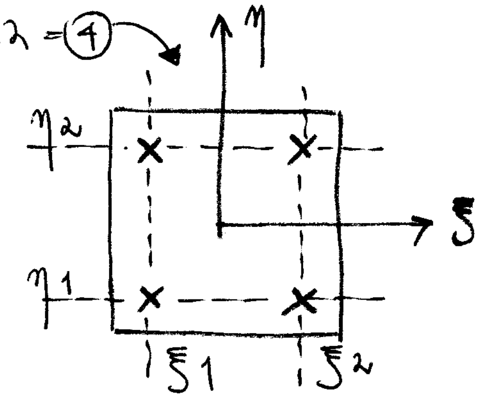
$$= \int_{-1}^1 \frac{(d-c)(b-a)}{4}\, d\xi\, d\eta = \sum_{i=1}^{1} \sum_{j=1}^{1} \frac{(d-c)(b-a)}{4} \cdot w_i \cdot w_j$$

$$= \frac{(d-c)(b-a)}{\cancel{4}} \cdot \cancel{2} \cdot \cancel{2} = \underline{\underline{(b-a)(d-c)}}$$

→ PRIMER: $\int_a^b \int_c^d x^2 y^2 \, dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \left(\frac{b-a}{2}\xi + \frac{b+a}{2}\right)^2 \cdot$

$\cdot \left(\frac{d-c}{2}\eta + \frac{d+c}{2}\right)^2 \frac{(b-a)(d-c)}{4} \, d\xi\, d\eta = *$

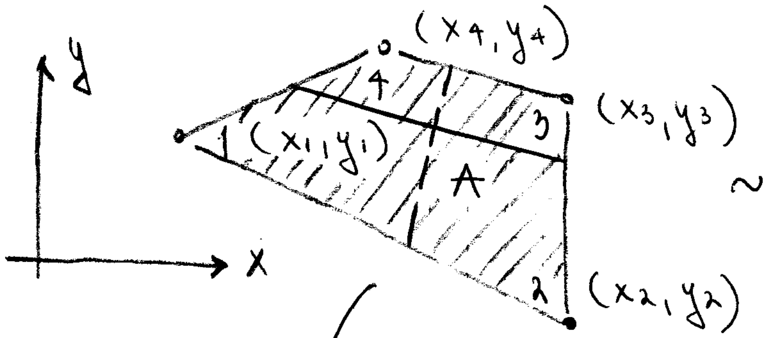
$*)$ 2 INT. TOČKI V ξ $\Big\}$ $2 \times 2 = 4$
2 INT. TOČKI V η



$* = \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j f(\xi_i, \eta_j) =$

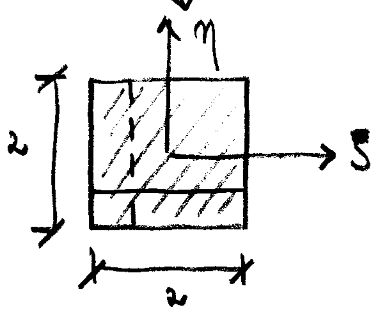
$= \underset{1}{w_1}\,\underset{1}{w_1}\, f(-\sqrt{3}/3, -\sqrt{3}/3) +$
$+ \underset{1}{w_1}\,\underset{1}{w_2}\, f(-\sqrt{3}/3, \sqrt{3}/3) +$
$+ \underset{1}{w_2}\,\underset{1}{w_1}\, f(\sqrt{3}/3, -\sqrt{3}/3) +$
$+ \underset{1}{w_2}\,\underset{1}{w_2}\, f(\sqrt{3}/3, \sqrt{3}/3)$

D.7. INTEGRACIJA PO POLJUBNEM ČETVEROKOTNIKU



$\sim \int_A f(x,y)\,dx\,dy$

$\left.\begin{array}{l} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{array}\right\} \underset{\sim}{J}$



→ KAKO DOLOČITI $x = x(\xi,\eta)$ IN $y = y(\xi,\eta)$?

LAHKO UPORABIM LAGRANGEVO INTERPOLACIJO...

2013/14 1. URŠKA BAČAR NM. 2/B 18

LAGRANGEVA INTERPOLACIJA: $x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\, x_i$

$$y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\, y_i$$

$$N_i = \begin{cases} 1, & \text{V VOZLIŠČU } i \\ 0, & \text{V OSTALIH VOZLIŠČIH} \end{cases}$$

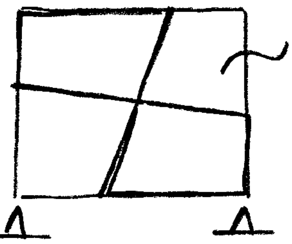
$$N_1 = 1/4\ (\xi-1)(\eta-1)$$
$$N_2 = -1/4\ (\xi+1)(\eta-1)$$
$$N_3 = 1/4\ (\xi+1)(\eta+1)$$
$$N_4 = -1/4\ (\xi-1)(\eta+1)$$

$$\underset{\sim}{J} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\eta \\ \partial y/\partial\xi & \partial y/\partial\eta \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i \\ \sum_{i=1}^{4} \frac{\partial N}{\partial \xi} y_i & \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \Rightarrow \det \underset{\sim}{J} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$\int_A f(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \underbrace{f(x(\xi,\eta)\ y(\xi,\eta))\ \det \underset{\sim}{J}(\xi,\eta)}_{p(\xi,\eta)}\, d\xi\, d\eta =$$

STENA S KE:



$$\int_A f(\ )\,dx\,dy = \underset{\sim}{K}$$

$$= \sum_{i=1}^{Ng\xi} \sum_{j=1}^{Ng\eta} w_i\, w_j\ p(\xi,\eta)$$

$x = x(\xi,\eta) = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)x_4 = \sum_{i=1}^{4} N_i(\xi,\eta)\,x_i$

NA ROBU DOBIMO PRAVE VREDNOSTI, VMES PA INTERPOLIRAMO

$y = y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\,y_i$

Lagrangeve interpolacijske funkcije: $N_i(\xi,\eta) = \begin{cases} 1, & \text{v vozlišču } i \\ 0, & \text{v vseh ostalih vozliščih} \end{cases}$

$N_1 = +\frac{1}{4}(\xi-1)(\eta-1)$

$N_2 = -\frac{1}{4}(\xi+1)(\eta-1)$

$N_3 = \frac{1}{4}(\xi+1)(\eta+1)$

$N_4 = -\frac{1}{4}(\xi-1)(\eta+1)$

$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \xi} x_i \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \eta} x_i$

$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \xi} y_i \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i(\xi,\eta)}{\partial \eta} y_i$

$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(\eta-1) \qquad \frac{\partial N_2}{\partial \xi} = -\frac{1}{4}(\eta-1) \qquad \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(\eta+1) \qquad \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(\eta+1)$

$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(\xi-1) \qquad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(\xi+1) \qquad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(\xi+1) \qquad \frac{\partial N_4}{\partial \eta} = -\frac{1}{4}(\xi-1)$

$\rightarrow \frac{\partial x}{\partial \xi} = \frac{1}{4}(\eta-1)\overset{0}{x_1} - \frac{1}{4}(\eta-1)\overset{4}{x_2} + \frac{1}{4}(\eta+1)\overset{3}{x_3} - \frac{1}{4}(\eta+1)\overset{1}{x_4} = \frac{1}{4}(3\eta+3-\eta-1) - \frac{1}{4}(4\eta-4) = \frac{1}{4}(-2\eta+6)$

$\rightarrow \frac{\partial x}{\partial \eta} = \frac{1}{4}(\xi-1)\overset{0}{x_1} - \frac{1}{4}(\xi+1)\overset{4}{x_2} + \frac{1}{4}(\xi+1)\overset{3}{x_3} - \frac{1}{4}(\xi-1)\overset{1}{x_4} = \frac{1}{4}(-4\xi-4+3\xi+3) - \frac{1}{4}(\xi-1) = \frac{1}{4}(-2\xi)$

$\rightarrow \frac{\partial y}{\partial \xi} = \frac{1}{4}(\eta-1)\overset{0}{y_1} - \frac{1}{4}(\eta-1)\overset{1}{y_2} + \frac{1}{4}(\eta+1)\overset{4}{y_3} - \frac{1}{4}(\eta+1)\overset{3}{y_4} = \frac{1}{4}(-\eta+1+4\eta+4-3\eta-3) = +\frac{2}{4}$

$\rightarrow \frac{\partial y}{\partial \eta} = \frac{1}{4}(\xi-1)\overset{0}{y_1} - \frac{1}{4}(\xi+1)\overset{1}{y_2} + \frac{1}{4}(\xi+1)\overset{4}{y_3} - \frac{1}{4}(\xi-1)\overset{3}{y_4} = \frac{1}{4}(-\xi-1+4\xi+4-3\xi+3) = \frac{1}{4}$

$\underset{\sim}{J} = \begin{bmatrix} \frac{1}{2}(\eta+3) & -\frac{\xi}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \qquad \det \underset{\sim}{J} = \frac{3}{4}(-\eta+3) + \frac{\xi}{4}$

$\sum_{i=1}^{1}\sum_{j=1}^{1} w_i w_j f(\xi_i,\eta_i)$

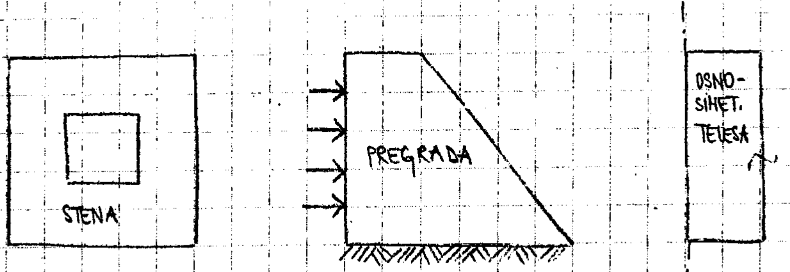
$\int_{-1}^{1}\int_{-1}^{1} \det \underset{\sim}{J}\, d\xi\, d\eta = \int_{-1}^{1}\int_{-1}^{1} \left[\frac{3}{4}(-\eta+3) + \frac{\xi}{4}\right] d\xi\, d\eta = \overset{2}{w_1}\,\overset{2}{w_2}\, f(\underbrace{\xi=0}_{\xi_1}, \underbrace{\eta=0}_{\eta_1}) = 2\cdot 2\cdot \frac{9}{4} = 9$

Koliko točk potrebujemo za točno numerično integracijo?

$\left.\begin{matrix} \xi \ldots 1 \\ \eta \ldots 1 \end{matrix}\right\} 1\times 1 = 1$ točke: $w_1 = 2,\ \xi_1 = \eta_1 = 0$

C. KONČNI ELEMENTI ZA RAVNINSKE PLOSKOVNE KONSTRUKCIJE

PRIMERI:



značilno za ravnino, tunel ~ ASolid

Konstrukcije v RNS (ravninsko nap. stanje) (vse napetosti, ki imajo indeks debeline so zanemarljive) ~ Plane Stress

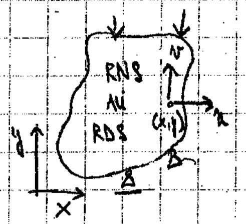
Konstrukcije v RDS (ravninsko def. stanje) ~ Plane Strain

ker smer je zelo dolga in zato tukaj zanemarimo deformacije

C.1. ENAČBE ZA RNS IN RDS

Osnovne enačbe vsakega konstrukcijskega modela so 3 vrste:
- kinematične,
- konstitutivne,
- ravnotežne.

\+ ROBNI POGOJI



KINEMATIČNE ENAČBE:

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx}(x,y) \\ \varepsilon_{yy}(x,y) \\ \gamma_{xy}(x,y) = 2\varepsilon_{xy}(x,y) \end{Bmatrix}$$

γ_{xy} – INŽENIRSKA STRIŽNA DEFORMACIJA; ε_{xy} – KOMPONENTA TENZORJA DEF.

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \frac{\partial u(x,y)}{\partial x} \\ \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\underset{\sim}{L}} \underbrace{\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}}_{\underset{\sim}{u}} \rightarrow \boxed{\underset{\sim}{\varepsilon} = \underset{\sim}{L}\,\underset{\sim}{u}} \quad (1)$$

KONSTITUTIVNE ENAČBE:

$$\boxed{\underset{\sim}{\sigma} = \underset{\sim}{E} \cdot \underset{\sim}{\varepsilon}} \quad (2)$$

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{Bmatrix}$$

E – ELASTIČNI MODUL, ν – POISSONOV KOLIČNIK

$$\underset{\sim}{E} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \quad \text{(RNS)}$$

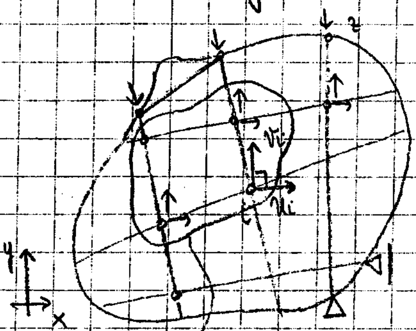
$$\underset{\sim}{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad \text{(RDS)}$$

RAVNOTEŽNE ENAČBE:

$$\left.\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+v_x&=0\\ \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+v_y&=0\end{aligned}\right\}\quad \underbrace{\begin{bmatrix}\frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}}_{\underset{\sim}{L}^T}\underbrace{\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}}_{\underset{\sim}{\sigma}}+\underbrace{\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}}_{\underset{\sim}{v}}=\underbrace{\begin{Bmatrix}0\\ 0\end{Bmatrix}}_{\underset{\sim}{0}}\rightarrow \boxed{\underset{\sim}{L}^T\underset{\sim}{\sigma}+\underset{\sim}{v}=\underset{\sim}{0}}\quad (3)$$

VOLUMSKI SILI $[N/m^3]$

Te tri notranje enačbe morajo biti zadoščene, ko računamo z metodo MKE!



Pri MKE: $\underset{\sim}{F}=\underset{\sim}{K}\cdot\underset{\sim}{u}$

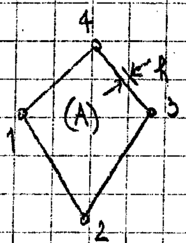
VEKTOR VOZLIŠČNIH POMIKOV

VEKTOR VOZLIŠČNIH OBTEŽB

TOGOSTNA MATRIKA KONST.

Kako to dosežemo? To dosežemo tako, da jih uporabimo za vse KE v mreži posebej, da določimo togostno matriko in obtežne sile v vozliščih. Poglejmo, kako določimo togostno mat. MKE za RNS ali RDS.

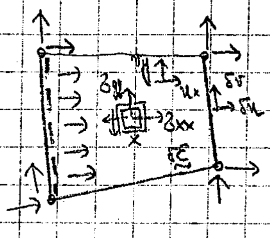
Startamo s (3) in enačbo preredimo na šibko variacijsko obliko, ki jo poznamo kot princip virtualnega dela.



PVD: $$\underbrace{h\int_A \delta\underset{\sim}{\varepsilon}^T\underset{\sim}{\sigma}\,dA}_{\text{VIRTUALNO DELO "NOT. SIL"}}=\underbrace{h\int_A \delta\underset{\sim}{u}^T\underset{\sim}{v}\,dA+\int_\Gamma \delta\underset{\sim}{u}^T\underset{\sim}{t}\,d\Gamma}_{\text{VIRTUALNO DELO "ZUN. SIL"}}+\text{SILE V VOZLIŠČIH}$$

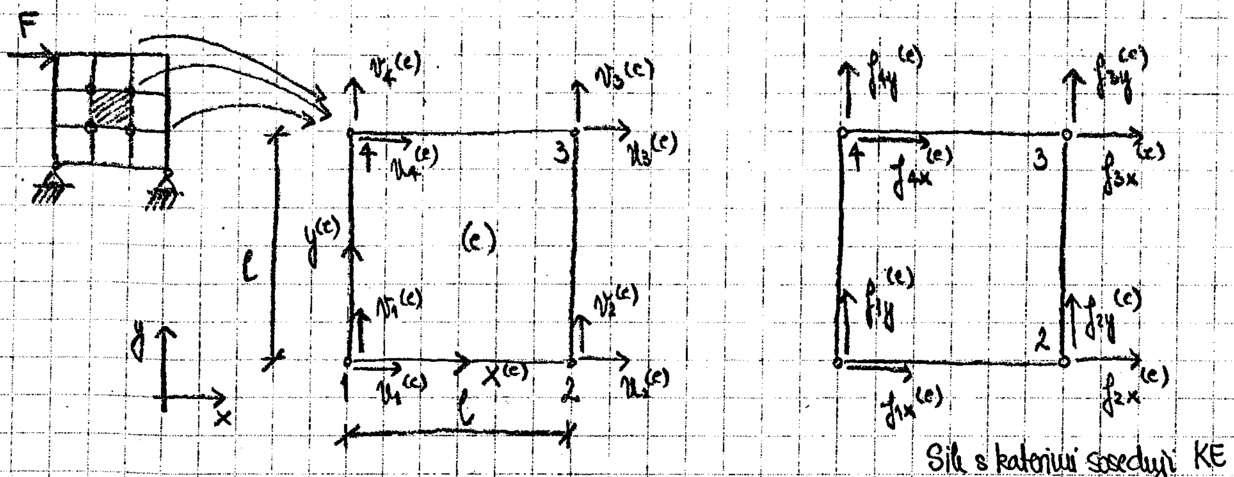
h = DEBELINA

$$\delta\underset{\sim}{\varepsilon}=\begin{Bmatrix}\delta\varepsilon_{xx}(x,y)\\ \delta\varepsilon_{yy}(x,y)\\ \delta\gamma_{xy}(x,y)\end{Bmatrix}\ \ldots\ \text{VIRTUALNE DEFORMACIJE}$$



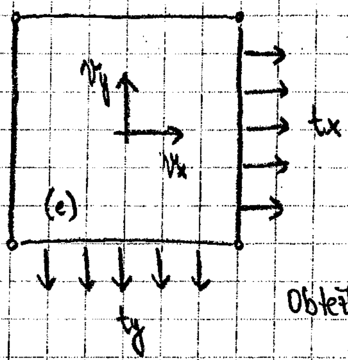
$$\delta\underset{\sim}{\varepsilon}=\begin{Bmatrix}\frac{\partial\delta u(x,y)}{\partial x}\\ \frac{\partial\delta v(x,y)}{\partial y}\\ \frac{\partial\delta u(x,y)}{\partial y}+\frac{\partial\delta v(x,y)}{\partial x}\end{Bmatrix}$$

C.2. KVADRATNI KE ZA RNS IN RDS



Pomiki v različnih KE.

Sile s katerimi sosednji KE delujejo na (e).



Obtežba, ki deluje na (e).

Če je KE(e) v ravnotežju (se ne premika). Potem mora biti zanj izpolnjena enačba virtualnega dela. Enačba virtualnega dela je šibka oblika ravnotežne enačbe. Če v to enačbo vstavimo še konstitutivne enačbe in kinematične enačbe, dobimo:

$$\underbrace{\underset{\text{DEBELINA}}{h^{(e)}}\int_{A^{(e)}} \underset{\text{VIRTUALNE DEF.}}{\delta\underset{\sim}{\varepsilon}^T}\, \underset{\text{NAPETOST}}{\underset{\sim}{\sigma}}\, dA}_{\text{VIRTUALNO DELO NOTRANJIH SIL}} = h^{(e)}\int_{A^{(e)}} \underset{\text{VIRT. POMIKI}}{\delta\underset{\sim}{u}^{(e)T}}\, \underset{\text{VOLUMSKE SILE}}{\underset{\sim}{v}}\, dA + \int_{\Gamma^{(e)}} \delta\underset{\sim}{u}^T \underset{\text{SILE NA ROBU KE}}{\underset{\sim}{t}}\, ds + \ldots$$

ENAČBA NEVTRALNEGA DELA

VIRTUALNO DELO ZUNANJIH SIL

$$+\ \delta u_1^{(e)} f_{1x}^{(e)} + \delta v_1^{(e)} f_{1y}^{(e)} + \ldots + \delta u_4 f_{4x}^{(e)} + \delta v_4^{(e)} f_{4y}^{(e)}$$

KINEMATIČNE ENAČBE: $\underset{\sim}{\varepsilon} = \underset{\sim}{L}\cdot\underset{\sim}{u}^{(e)}$

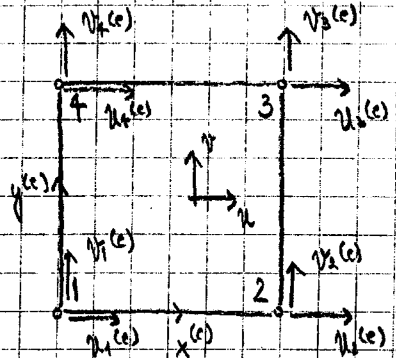
$\delta\underset{\sim}{\varepsilon} = \underset{\sim}{L}\,\delta\underset{\sim}{u}^{(e)}$

KONSTITUTIVNA ENAČBA: $\underset{\sim}{\sigma} = \underset{\sim}{E}\cdot\underset{\sim}{\varepsilon}$

$\underset{\sim}{\sigma} = \underset{\sim}{E}\cdot\underset{\sim}{L}\cdot\underset{\sim}{u}^{(e)}$

$$h^{(e)}\int_{A^{(e)}} (\delta u^{(e)})^T L^T E L u^{(e)}\, dA = h^{(e)}\int_{A^{(e)}} (\delta u^{(e)})^T v\, dA + \int_{\Gamma^{(e)}} (\delta u^{(e)})^T t\, ds + \delta u_1^{(e)} f_{1x}^{(e)} \dots \quad (*)$$

V gornji enačbi je neznanka edino $u^{(e)}$, ker je $\delta u^{(e)}$ poljuben. Če hočem kaj izračunati, moram vpeljati interpolacijo pomikov!



$$u(x^{(e)}, y^{(e)}) = \sum_{i=1}^{4} N_i(x^{(e)}, y^{(e)})\, u_i^{(e)}$$

$$v(x^{(e)}, y^{(e)}) = \sum_{i=1}^{4} N_i(x^{(e)}, y^{(e)})\, v_i^{(e)}$$

$$N_i(x^{(e)}, y^{(e)}) = \begin{cases} 1 & \text{v vozlišču } i \\ 0 & \text{v ostalih vozliščih} \end{cases}$$

$$N_1(x^{(e)}, y^{(e)}) = \frac{1}{l^2}(x^{(e)} - l)(y^{(e)} - l)$$

$$N_2(x^{(e)}, y^{(e)}) = -\frac{1}{l^2}(x^{(e)})(y^{(e)} - l)$$

$$N_3(x^{(e)}, y^{(e)}) = \frac{1}{l^2}(x^{(e)})(y^{(e)})$$

$$N_4(x^{(e)}, y^{(e)}) = -\frac{1}{l^2}(x^{(e)} - l)(y^{(e)})$$

$$u^{(e)} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \underbrace{\begin{Bmatrix} u_1^{(e)} \\ v_1^{(e)} \\ u_2^{(e)} \\ v_2^{(e)} \\ u_3^{(e)} \\ v_3^{(e)} \\ u_4^{(e)} \\ v_4^{(e)} \end{Bmatrix}}_{p^{(e)}}$$

$$\boxed{u^{(e)} = N \cdot p^{(e)}}$$

Enako interpolacijo uporabimo tudi za virtualna pomika δu, δv. Torej velja:

$$\delta u^{(e)} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix} = N\, \delta p^{(e)}$$

$$\delta p^{(e)} = \begin{Bmatrix} \delta u_1^{(e)} \\ \delta v_1^{(e)} \\ \vdots \\ \delta u_4^{(e)} \\ \delta v_4^{(e)} \end{Bmatrix}$$

$$\boxed{\delta u^{(e)} = N \cdot \delta p^{(e)}}$$

Vstavimo $u^{(e)} = N \cdot p^{(e)}$ in $\delta u^{(e)} = N\, \delta p^{(e)}$ v enačbo $(*)$ in dobimo:

$$\underset{\text{POLJUBEN}}{\delta p^{(e)}} \Big[\underbrace{h^{(e)} \int_{A^{(e)}} (\overbrace{L N}^{B})^T E (\overbrace{L N}^{B})\, dA}_{K^{(e)}}\, u^{(e)} - \underbrace{h^{(e)} \int_{A^{(e)}} N^T v^{(e,V)}\, dA - \int_{\Gamma^{(e)}} N^T t\, ds}_{F_{OBT}^{(e)}} - f^{(e)} \Big] = 0$$

Dobimo:

$$\boxed{\underset{\sim}{K}^{(e)}\underset{\sim}{u}^{(e)} = \underset{\sim}{f}^{(e)} + \underset{\sim}{F}_{OBT}^{(e)}}$$

Togostno matrika $K^{(e)}$ je torej: $\underset{\sim}{K}^{(e)} = h^{(e)}\int_{A^{(e)}} \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA$

Obtežni vektor KE je:

$$\boxed{\underset{\sim}{F}_{OBT}^{(e)} = h^{(e)}\int_{A^{(e)}} \underset{\sim}{N}^T \underset{\sim}{v}\, dA + \int_{\Gamma^{(e)}} \underset{\sim}{N}^{(T)} \underset{\sim}{t}\, ds}$$

$[kN/m]$

LASTNA TEŽA

$$\underset{\sim}{v} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\gamma \end{Bmatrix}$$

Zdaj imamo obtežni vektor in togostno matriko elementa (e). Togostno matriko konstrukcije dobimo iz pogoja, da so vsa različna mrežna KE v ravnotežju. V „praksi" to pomeni, da togostne matrike vseh končnih elementov „sestavimo" na pravilen način v togostno matriko konstrukcije. Enako velja za obtežni vektor.

$$\boxed{\underset{\sim}{F}_{OBT} = \underset{\sim}{K} \cdot \underset{\sim}{u}}$$

Določimo togostno matriko $K^{(e)}$ za poissonov količnik $\nu = 0$!

Najprej poglejmo $\underset{\sim}{B}$:

$$\underset{\sim}{B} = \left.\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}_{3\times 2} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}_{2\times 8}\right|_{\substack{x=x^{(e)} \\ y=y^{(e)}}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{1}{\ell^2}(y-\ell) & \cdots \\ 0 & \cdots \\ \frac{1}{\ell^2}(x-\ell) & \cdots \end{bmatrix}$$

Poglejmo si še $\left.\underset{\sim}{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|_{\nu=0}$

RNS=RDS

Či vrednim integral:

$$h^{(e)}\int_0^\ell\int_0^\ell \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dx\, dy = \frac{Eh^{(e)}}{8}\begin{bmatrix} 4 & 1 & -2 & -1 & -2 & -1 & 0 & 1 \\ 1 & 4 & 1 & 0 & -1 & -2 & -1 & -2 \\ -2 & 1 & 4 & -1 & 0 & -1 & -2 & 1 \\ -1 & 0 & -1 & 4 & 1 & -2 & 1 & -2 \\ -2 & -1 & 0 & 1 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & -2 & 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 & -2 & 1 & 4 & -1 \\ 1 & -2 & 1 & -2 & -1 & 0 & -1 & 4 \end{bmatrix}$$

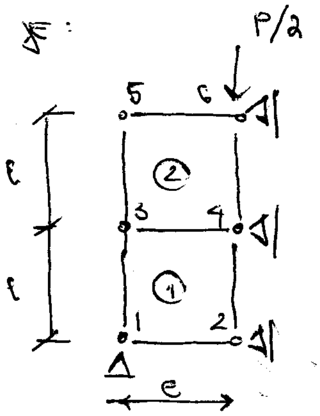
C.3. RAČUNSKI PRIMER

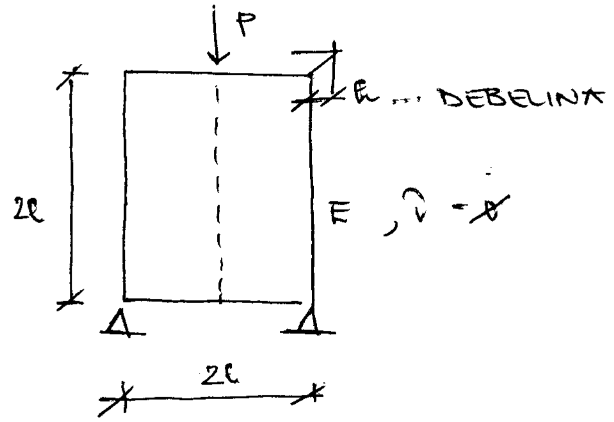
ENAČBA, KI JO BOMO REŠEVALI ZA MREŽO:

$\boxed{\underset{\sim}{F} = \underset{\sim}{k} \cdot \underset{\sim}{d}}$ — TOGOSTNA MATRIKA KONS.

OBTEŽNI VEKTOR KONSTRUKCIJE

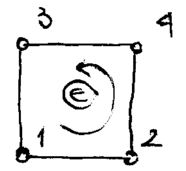
VEKTOR VOZLIŠČNIH POMIKOV KONST.



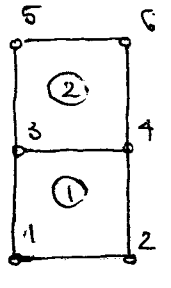


*) SAP: PLANE, PLANE STRESS
BREZ "INCOMPATIBLE MODEL"

$$\underset{\sim}{F} = \begin{array}{c|l} \left\{\begin{array}{c} F_{1x} \\ F_{1y} \\ F_{2x} \\ 0 \\ F_{3x} \\ 0 \\ 0 \\ 0 \\ F_{5x} \\ -P/2 \\ 0 \\ 0 \end{array}\right\} & \begin{array}{l} \\ \\ \\ 4 \\ \\ 6 \\ 7 \\ 8 \\ \\ 10 \\ 11 \\ 12 \end{array} \end{array} \qquad \underset{\sim}{d} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ v_2 \\ 0 \\ v_3 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ u_6 \\ v_6 \end{array}\right\}$$



$$\underset{\sim}{k}^{(e)} = \frac{E^{(e)} h^{(e)}}{8} \begin{array}{c} \left[\begin{array}{cc|cc|cc|cc} 4 & 1 & -2 & -1 & -2 & -1 & 0 & 1 \\ 1 & 4 & 1 & 0 & -1 & -2 & -1 & 2 \\ 2 & 1 & 4 & -1 & 0 & -1 & -2 & 1 \\ -1 & 0 & -1 & 4 & 1 & -2 & 1 & -2 \\ -2 & -1 & 0 & 1 & 4 & 1 & -2 & 1 \\ -1 & -2 & -1 & -2 & 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 & -2 & 1 & 4 & -1 \\ 1 & -2 & 1 & -2 & -1 & 0 & -1 & 4 \end{array}\right] \\ \begin{array}{cccc} \underbrace{\quad\;\;}_{1} & \underbrace{\quad\;\;}_{2} & \underbrace{\quad\;\;}_{3} & \underbrace{\quad\;\;}_{4} \end{array} \end{array} \begin{array}{l} \left.\right\} 1 \\ \\ \left.\right\} 2 \\ \\ \left.\right\} 3 \\ \\ \left.\right\} 4 \\ \\ \end{array}$$



GLOBALNE OZNAKE

2013/14 1. URŠKA BAČAR NM. 2/C 2G

$$\underset{\sim}{K} = \frac{EA}{8}$$

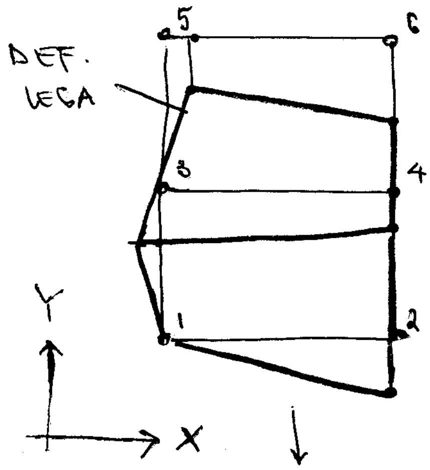
$$
\begin{array}{c|cc|cc|cc|cc|cc|cc|}
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & \\
\hline
1 & 4 & 1 & -2 & -1 & -2 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & 4 & 1 & 0 & -1 & -2 & -1 & 2 & 0 & 0 & 0 & 0 \\
\hline
2 & -2 & 1 & 4 & -1 & 0 & -1 & -2 & 1 & 0 & 0 & 0 & 0 \\
 & -1 & 0 & -1 & 4 & 1 & -2 & 1 & -2 & 0 & 0 & 0 & 0 \\
\hline
3 & -2 & -1 & 0 & 1 & 4+4 & 1-1 & -2-2 & 1+1 & 0 & -1 & -2 & 1 \\
 & -1 & -2 & -1 & -2 & 1-1 & 4+4 & 1-1 & 0+0 & 1 & -2 & 1 & -2 \\
\hline
4 & 0 & -1 & -2 & 1 & -2-2 & 1-1 & 4+4 & -1+1 & -2 & -1 & 0 & 1 \\
 & 1 & -2 & 1 & -2 & -1+1 & 0+0 & -1+1 & 4+4 & -1 & -2 & -1 & -2 \\
\hline
5 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -1 & 4 & 1 & -2 & 1 \\
 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -2 & 1 & 4 & 1 & 0 \\
\hline
6 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & -1 & -2 & 1 & 4 & -1 \\
 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -2 & -1 & 0 & -1 & 4 \\
\hline
\end{array}
$$

IZRAČUNAMO NEZNANE POMIKE TAKO, DA VZAMEMO SAMO TISTE ENAČBE IZ $\underset{\sim}{F} = \underset{\sim}{K}\,\underset{\sim}{d}$, KI SE NANAŠAJO NA ZNANE SILE:

$$
\begin{array}{c}
4\\ 6\\ 7\\ 8\\ 10\\ 11\\ 12
\end{array}
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -P/2 \\ 0 \\ 0 \end{Bmatrix}
= \frac{EA}{8}
\begin{bmatrix}
4 & -2 & 1 & -2 & 0 & 0 & 0 \\
-2 & 8 & 0 & 0 & -2 & 1 & -2 \\
1 & 0 & 8 & 0 & -1 & 0 & 1 \\
-2 & 0 & 0 & 8 & -2 & -1 & -2 \\
0 & -2 & -1 & -2 & 4 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 & 4 & -1 \\
0 & -2 & 1 & -2 & 0 & -1 & 4
\end{bmatrix}
\begin{Bmatrix} v_2 \\ v_3 \\ u_4 \\ v_4 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}
$$

IZ REDUCIRANEGA SISTEMA LAHKO IZRAČUNAMO NEZNANE POMIKE:

$$
\begin{Bmatrix} v_2 \\ v_3 \\ u_4 \\ v_4 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}
= -\frac{P}{EA}
\begin{Bmatrix} 89/90 \\ 31/30 \\ 1/45 \\ 43/45 \\ 189/90 \\ -14/45 \\ 44/45 \end{Bmatrix}
$$



SLABI REZULTATI, KER JE MREŽA PREGROBA.

KO POZNAMO POMIKE, LAHKO IZRAČUNAMO REAKCIJE IZ $\underset{\sim}{F} = \underset{\sim}{K}\,\underset{\sim}{d}$.

VZAMEMO SAMO TISTE ENAČBE, V KATERIH SO REAKCIJE NEZNANE.

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{3x} \\ F_{5x} \end{Bmatrix} = \frac{EL}{8}\left(-\frac{P}{EL}\right) \begin{bmatrix} -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & -2 & 0 & 0 & 0 \\ -1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & -4 & 2 & -1 & -2 & 1 \\ 0 & 1 & -2 & -1 & 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} 89/90 \\ 31/30 \\ 1/45 \\ 43/45 \\ 189/90 \\ -14/45 \\ 44/45 \end{Bmatrix}$$

KAKO IZRAČUNAMO NAP. PO ELEMENTIH?
TO NAREDIMO ZA VSAK ELEMENT POSEBAJ.

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}^{(e)} = \underset{\sim}{E}\,\underset{\sim}{B}\,\underset{\sim}{d}^{(e)}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \qquad \underset{\sim}{E} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \nu = 0$$

$$\underset{\sim}{d}^{(1)} = -\frac{P}{EL} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 89/90 \\ 0 \\ 31/30 \\ 1/45 \\ 43/45 \end{Bmatrix}$$

MORAM IZRAČUNAT ŠE $\underset{\sim}{B}(1)$ IN $\underset{\sim}{B}(2)$.

$$\underset{\sim}{d}^{(2)} = -\frac{P}{EL} \begin{Bmatrix} 1/45 \\ 43/45 \\ 0 \\ 31/30 \\ 0 \\ 189/90 \\ -14/45 \\ 44/45 \end{Bmatrix}$$

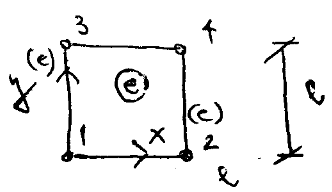
ZADNJIČ SMO NAPISALI:

$$\underset{\sim}{B}^{(e)} = \begin{bmatrix} \frac{\partial N_1}{\partial x^{(e)}} & 0 & \frac{\partial N_2}{\partial x^{(e)}} & 0 & \frac{\partial N_3}{\partial x^{(e)}} & 0 & \frac{\partial N_4}{\partial x^{(e)}} & 0 \\ 0 & \frac{\partial N_1}{\partial y^{(e)}} & 0 & \frac{\partial N_2}{\partial y^{(e)}} & 0 & \frac{\partial N_3}{\partial y^{(e)}} & 0 & \frac{\partial N_4}{\partial y^{(e)}} \\ \frac{\partial N_1}{\partial y^{(e)}} & \frac{\partial N_1}{\partial x^{(e)}} & \frac{\partial N_2}{\partial y^{(e)}} & \frac{\partial N_2}{\partial x^{(e)}} & \frac{\partial N_3}{\partial y^{(e)}} & \frac{\partial N_3}{\partial x^{(e)}} & \frac{\partial N_4}{\partial y^{(e)}} & \frac{\partial N_4}{\partial x^{(e)}} \end{bmatrix}$$

ČE NESEM ODVODE INT. FUNKCIJ V MATRIKO, DOBIMO:

$$\underset{\sim}{B}^{(e)} = \begin{bmatrix} -\frac{b_2}{L} & 0 & \frac{b_2}{L} & 0 & -\frac{y^{(e)}}{L^2} & 0 & -\frac{y^{(e)}}{L^2} & 0 \\ 0 & -\frac{b_1}{L} & 0 & -\frac{x^{(e)}}{L^2} & 0 & \frac{x^{(e)}}{L^2} & 0 & \frac{b_1}{L} \\ -\frac{b_1}{L} & -\frac{b_2}{L} & -\frac{x^{(e)}}{L^2} & \frac{b_2}{L} & \frac{x^{(e)}}{L^2} & \frac{y^{(e)}}{L^2} & \frac{b_1}{L} & -\frac{y^{(e)}}{L^2} \end{bmatrix}$$

$$b_1 = 1 - \frac{x^{(e)}}{L} \qquad b_2 = 1 - \frac{y^{(e)}}{L}$$



$N_1 = \frac{1}{\ell^2}\left(1 - x^{(e)}\right)\left(1 - y^{(e)}\right)$

$\vdots$

N_4

ZNAM TOREJ IZRAČUNATI TAKO $\underset{\sim}{B}^{(1)}$, $\underset{\sim}{B}^{(2)}$, KOT $\underset{\sim}{F}^{(1)} = \underset{\sim}{F}^{(2)} = \underset{\sim}{F}$, IN $\underset{\sim}{d}^{(1)}$ TER $\underset{\sim}{d}^{(2)}$.

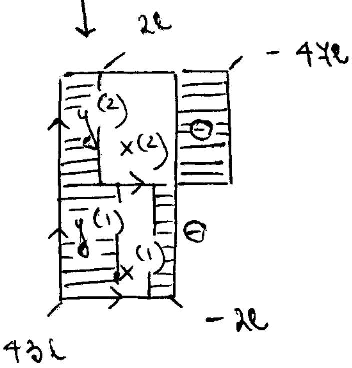
DOBIM NAPETOSTI:

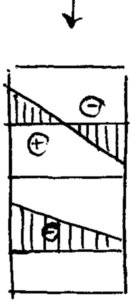
$$\begin{Bmatrix} \sigma_{xx}^{(1)} \\ \sigma_{yy}^{(1)} \\ \sigma_{xy}^{(1)} \end{Bmatrix} = \underset{\sim}{\sigma}^{(1)} = -\frac{p}{w\ell^2}\begin{Bmatrix} y^{(1)}/45 \\ (43\ell - 41x^{(1)})/45 \\ (87\ell + 2x^{(1)} - 82y^{(1)})/180 \end{Bmatrix}$$

SAP: AVERAGING ☒ YES ☐ NO

$$\begin{Bmatrix} \sigma_{xx}^{(2)} \\ \sigma_{yy}^{(2)} \\ \sigma_{xy}^{(2)} \end{Bmatrix} = \underset{\sim}{\sigma}^{(2)} = \frac{p}{wL^2}\begin{Bmatrix} -(8 + 13y^{(2)})/45 \\ (2\ell - 49x^{(2)})/45 \\ (19\ell - 26x^{(2)} - 89y^{(2)})/180 \end{Bmatrix}$$

$[\sigma_{yy}]\ \frac{p}{\ell^2}\cdot\frac{1}{45}$ $[\sigma_{yy}]$





VIDIMO DA SO $\sigma_{yy}^{(e)}$ PO ELEMENTU KONSTANTNE V SMERI $y^{(e)}$ IN LINEARNE PO $x^{(e)}$.

ZA $\sigma_{xx}^{(e)}$ VELJA, DA SO KONST. PO $x^{(e)}$ IN LINEARNE PO $y^{(e)}$.

$\sigma_{yy}^{(e)}$ SO LIN. PO $x^{(e)}$ IN $y^{(e)}$ PREKO ROBOV NAP. NISO ZVEZNE.

RAZLIKE POSTAJAJO MANJŠE Z BOLJ GOSTO MREŽO. KO ČITAMO REZULTA PONAVADI NAP. ZGLADIMO PREKO ROBOV ELEMENTOV.

6.4. IZOPARAMETRIČNI KE ZA RNS IN RDS

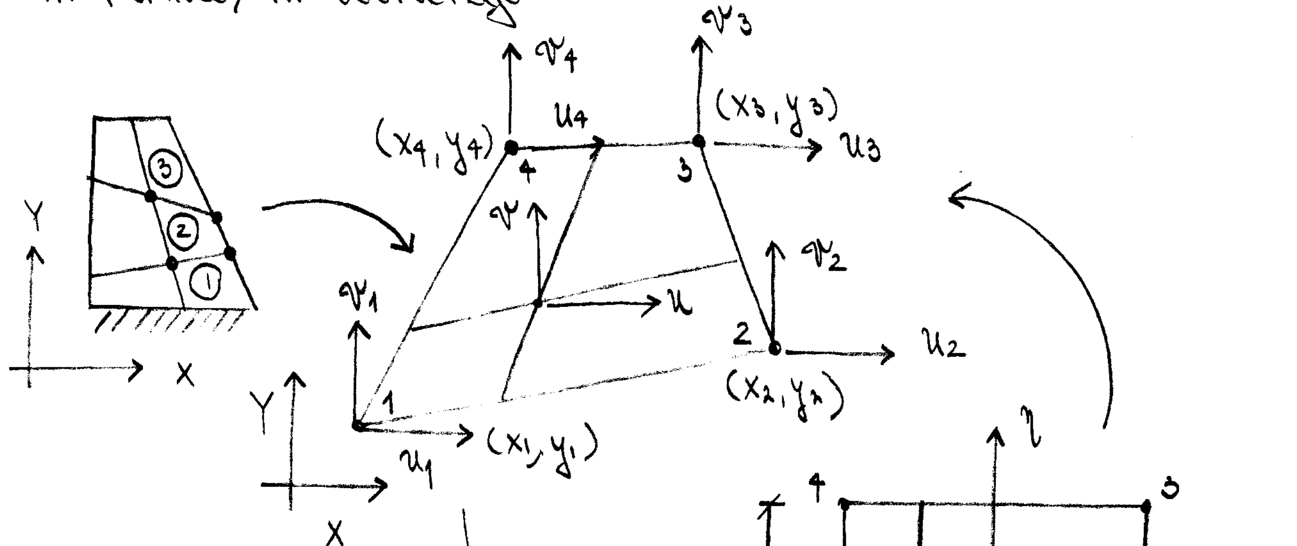
V PREJŠNJI TOČKI SMO SE UKVARJALI S KVADRATNIMI KE ZA RNS/RDS.

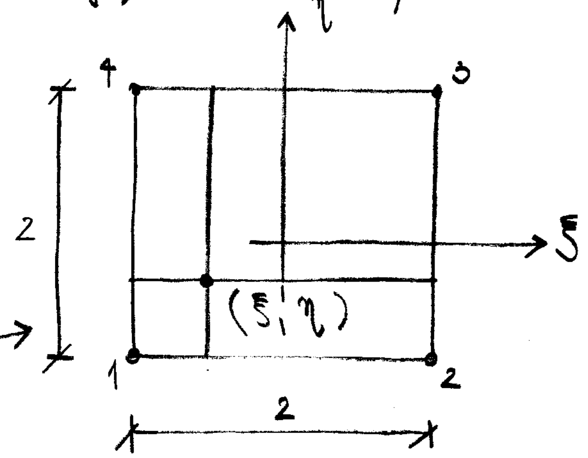
PONAVADI PA IMAMO MREŽO KE IZ PRAVOKOTNIKOV ALI NEPRAVILNIH ČETVEROKOTNIKOV.

IZOPARAMETRIČNI KE INTERPOLIRA NA ENAK NAČIN RAZLIČNE PARAMETRE.

LAGRANGEVE INT. F.

ZA POMIKE, ZA GEOMETRIJO





$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\, x_i$$

$$y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)\, y_i$$

LAGRANGEVE INT. F. NA $\square$:

$$N_1(\xi,\eta) = \frac{1}{4}(\xi-1)(\eta-1)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(\xi+1)(\eta-1)$$

$$N_3(\xi,\eta) = \frac{1}{4}(\xi+1)(\eta+1)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(\xi-1)(\eta+1)$$

$$u(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi_i,\eta_i)\, u_i$$

$$v(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi_i,\eta_i)\, v_i$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

KAKO DOLOČIM TOGOSTNO MATRIKO POVUBNEGA ČETVEROKOTNIKA.

IZ PRINCIPA VIR. DELA SLEDI, DA JE $\underset{\sim}{K}^{(e)} = \int_{A^{(e)}} \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dx\, dy$.

$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

RNS

$$\underset{\sim}{E} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix}$$

RDS

$$\underset{\sim}{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu/2 \end{bmatrix}$$

$$N_i = N_i(\xi, \eta) \Rightarrow \quad \frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

VIDIMO, DA VELJA:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial x/\partial \xi & \partial x/\partial \eta \\ \partial y/\partial \xi & \partial y/\partial \eta \end{bmatrix}}_{\underset{\sim}{J}} \underbrace{\begin{bmatrix} \partial \xi/\partial x & \partial \xi/\partial y \\ \partial \eta/\partial x & \partial \eta/\partial y \end{bmatrix}}_{\underset{\sim}{J}^{-1}}$$

$$= \begin{bmatrix} \frac{\partial x}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial x}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} & \frac{\partial x}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial x}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \\ \frac{\partial y}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} & \frac{\partial y}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial y}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$1 = \frac{dx}{\partial x} = \frac{\partial x}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial x}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$1 = \frac{\partial y}{\partial y} = \frac{\partial y}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial y}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$\cancel{0} = \frac{dx}{\partial y} = \frac{\partial x}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial x}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

ČLENE $\underset{\sim}{J}^{-1}$ DOBIMO TOREJ Z INTEGRIRANJEM $\underset{\sim}{J}$.

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underset{\sim}{A}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

KAKO INTEGRIRAMO $\underset{\sim}{k}^{(e)} = \int_{A^{(e)}} \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dx\,dy$.

SPREMENIMO INTEGRACIJSKO OBMOČJE

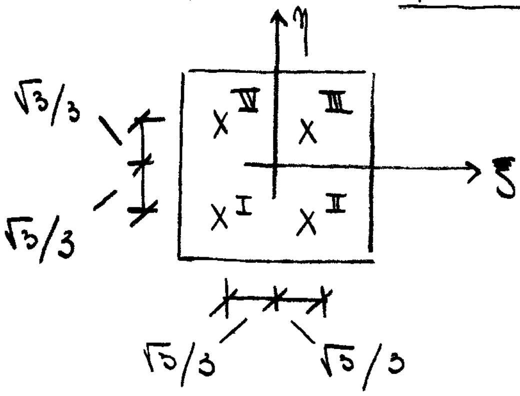
$$\int_{\square} \underbrace{\underset{\sim}{B}^T(\xi,\eta)\, \underset{\sim}{E}\, \underset{\sim}{B}(\xi,\eta)\, \det \underset{\sim}{J}(\xi,\eta)}_{\underset{\sim}{f}(\xi,\eta)}\, d\xi\, d\eta =$$

$$= w_1 w_1 \underset{\sim}{f}(-\sqrt{3}/3, -\sqrt{3}/3) + w_2 w_1 \underset{\sim}{f}(\sqrt{3}/3, -\sqrt{3}/3) +$$

$$+ w_1 w_2 \underset{\sim}{f}(-\sqrt{3}/3, \sqrt{3}/3) + w_2 w_2 \underset{\sim}{f}(\sqrt{3}/3, \sqrt{3}/3)$$

$w_1 = w_2 = 1$

NAREDIMO NUMERIČNO INTEGRACIJO:



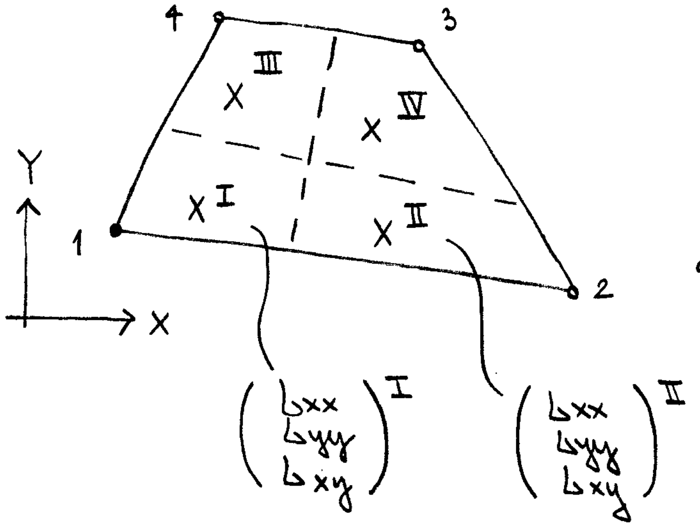
→ IZVREDNOTIMO $\underset{\sim}{B}$ IN $\det \underset{\sim}{J}$

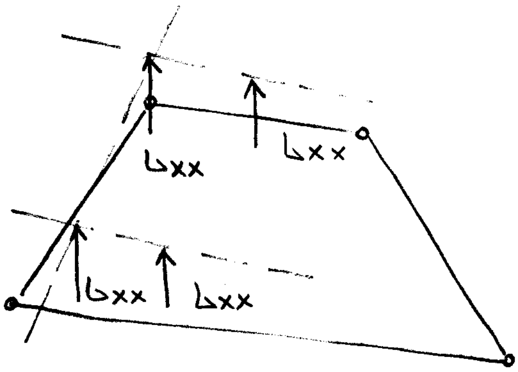
KO IZRAČUNAMO POMIKE V VOZLIŠČIH MREŽE LE LAHKO ZA VSAK ELEMENT MREŽE IZRAČUNAMO NAP

$$\underset{\sim}{\sigma}^{(e)} = \begin{Bmatrix} \sigma_{xx}^{(e)} \\ \sigma_{yy}^{(e)} \\ \sigma_{xy}^{(e)} \end{Bmatrix} = \underset{\sim}{E}\, \underset{\sim}{B}\, \underset{\sim}{d}^{(e)}$$

MATRIKO $\underset{\sim}{B}$ IMAMO IZRAČUNANO LE V INT. TOČKAH.

ZATO LAHKO IZRAČUNAMO TUDI NAP. V TEH TOČKAH. IZKAŽE SE, DA SO TO TOČKE, KJER ~~SO~~ NAP. PO ELEMENTU NAJBOLJ TOČNE.





ZA VREDNOSTI V INTEGRACIJSKIH TOČKAH DOBIM Z EKSTRAPOLACIJO VREDNOSTI PO CELEM KE.

SKOKI NAPETOSTI ČEZ ROB ELEMENTOV SE POJAVI ZATO, KER NE ZAHTEVAMO, DA SO TUDI ODVODI INTERPOLACIJSKIH FUNKCIJ ZVEZNI ČEZ ROB KE. TO ZVEZNOST ZAHTEVAMO SAMO ZA POMIKE.

PROGRAMI PO MKE TE SKOKE "ZGLADI" TAKO, DA GLEDAMO ZVEZNE NAP. (NOT. SILE) PO MREŽI KE.

6.5. KE ZA PLOŠČE

→ PROSOJNICE NA SPLETNI UČILNICI

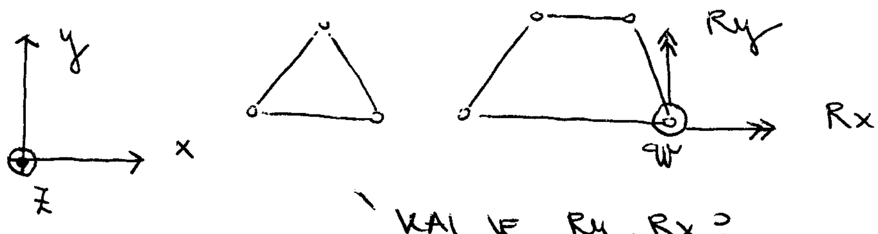
ZAKAJ SE UPORABLJAJO?

- SAMO ZA ANALIZO TAKŠNIH PLOŠČ, KJER SO OSNE SILE ZANEMARLJIVE:

  → OBTEŽBA PRAVOKOTNA NA PLOŠČO
  → GRE ZA RELATIVNO MASIVNE PLOŠČE
  → ČE SO PLOŠČE PRIMERNO PODPRTE

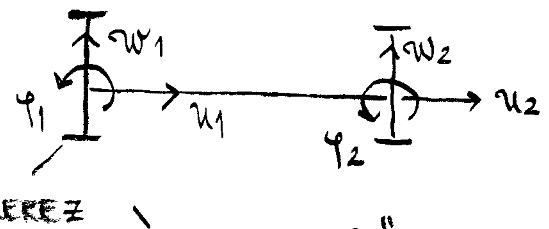
  } MEDETAŽNE AB (STROPNE) PLOŠČE

KE ZA PLOŠČE SO LAHKO:



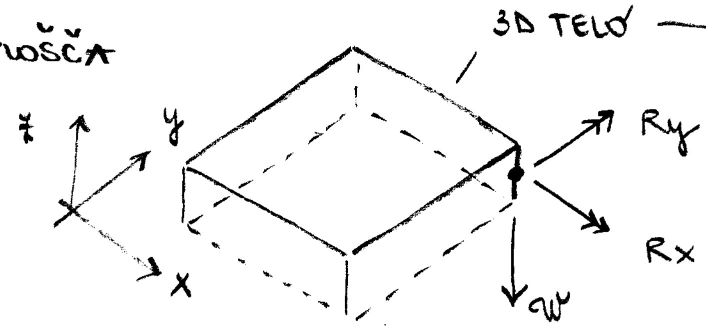
KAJ JE Ry, Rx ?

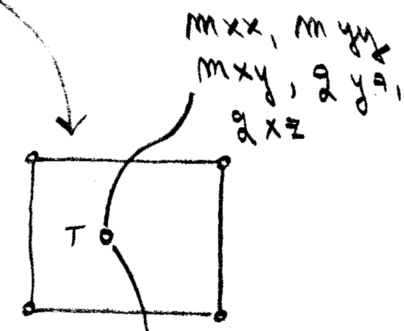
*) NOSILEC



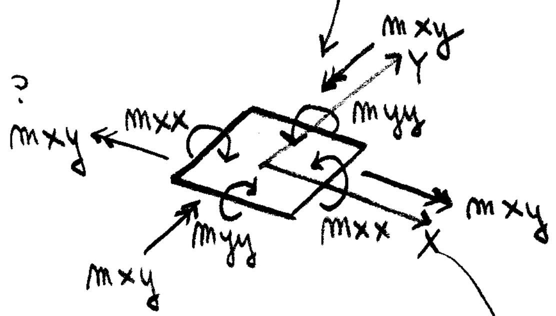
KAJ JE φ_1? "ZASUK PREREZA"

→ PLOŠČA





ROTACIJA SE POJAVI PRI ELEMENTIH, KJER SE REDUCIRA DEBELINO



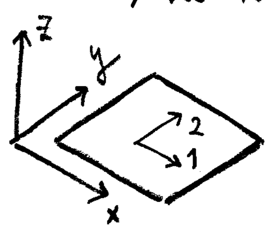
→ KAJ NAM IZRAČUNA TAKŠEN KE?

- POMIK w
- ROTACIJI R_x, R_y

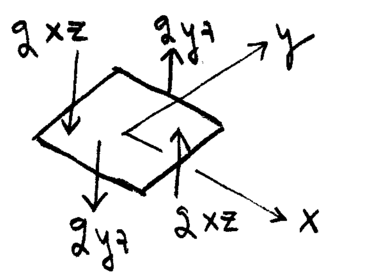
→ NOTRANJE SILE:

- MOMENTI: m_{xx}, m_{yy}, m_{xy}
- PREČNI SILI: q_{xz}, q_{yz} ⟶ KO RAČUNAM PREBOJ

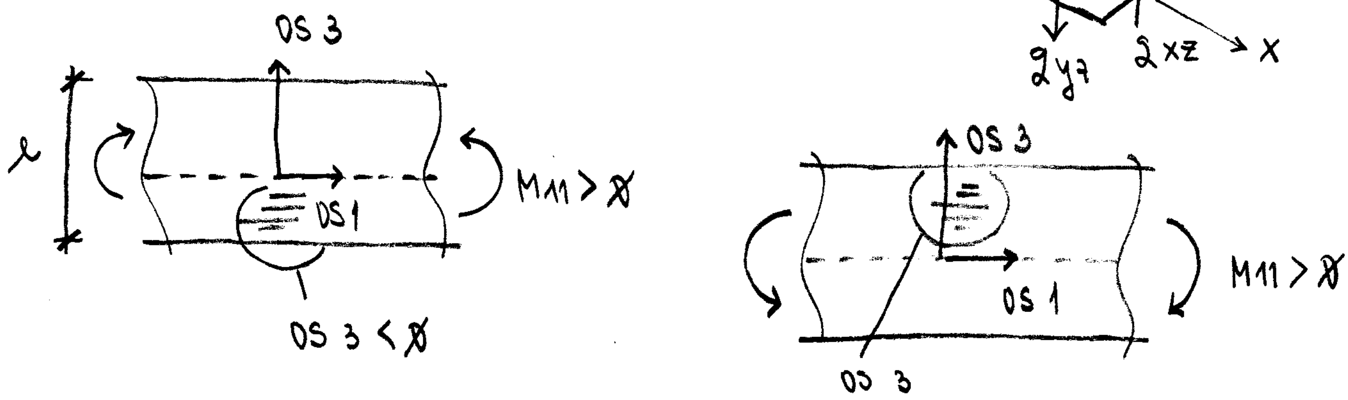
<u>SAP</u>: m_{xx} ... M11
m_{yy} ... M22 LOKALNI KS
m_{xy} ... M12



+
PREČNI SILI

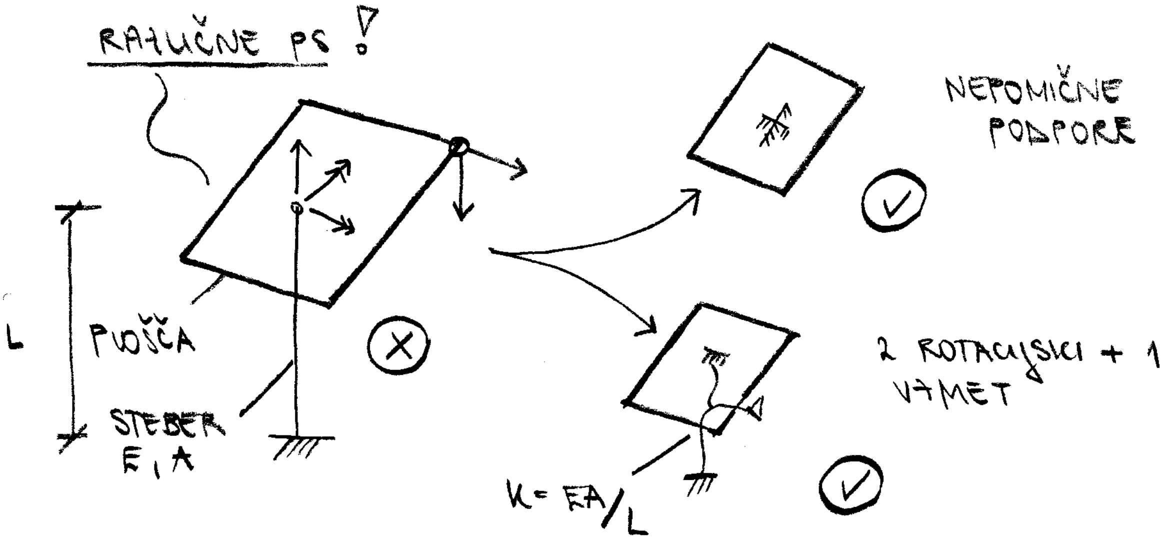


<u>PRAVILO</u>: POZITIVNI MOMENTI DAJEJO NATEGE NA TISTEM DELU PLOŠČE, KJER JE LOKALNA OS 3 NEGATIVNA.



PLOŠČE SO LAHKO PODPRTE NA RAZLIČNE NAČINE : S STEBRI, NOSILCI, STENAMI, ... :

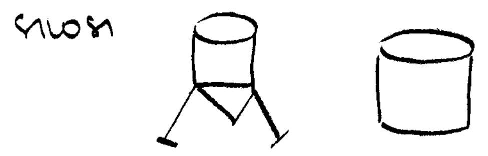
- ČE UPORABLJAMO KE ZA PLOŠČE, POTEM MORAMO STEBRE NADOMESTITI Z VZMETMI ALI NEPOMIČNIMI PODPORAMI
- NOSILCE MODELIRATI S KE ZA GRANO
- STENE NADOMESTITI Z VZMETMI ALI Z NEPOMIČNIMI PODPORAMI



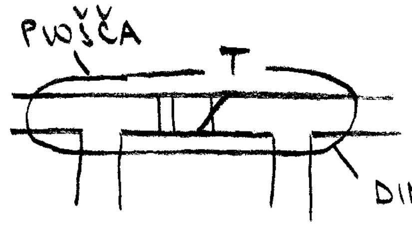
C.6. KE ZA LUPINE

→ KJE SE UPORABLJA?

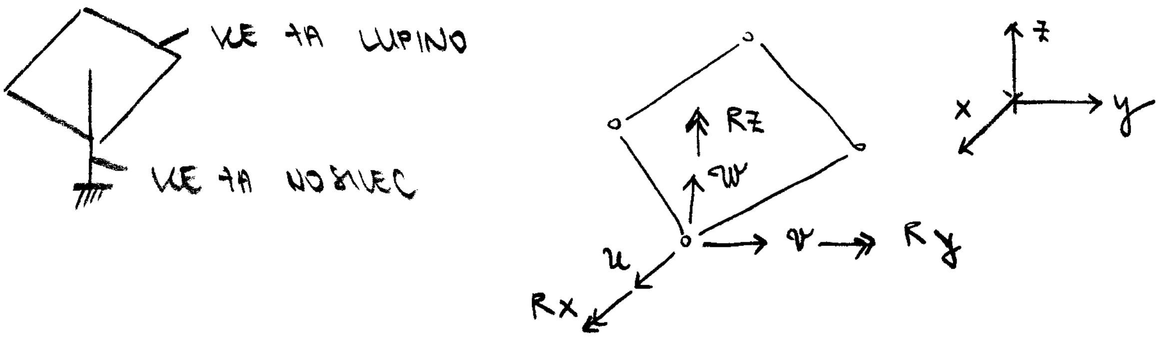
- ZA ANALIZO UKRIVLJENIH PLOSKOVNIH KONSTRUKCIJ



- ZA ANALIZO T.I. "RAVNIH LUPIN"
  ↳ PLOŠČE, PRI KATERIH SO OSNE SILE POMEMBNE



DIMENZIONIRATI S KE ZA LUPINO



→ ČE MODELIRAMO PLOŠČO Z KE ZA LUPINO, JO MORAMO USTREZNO PODPRET.